Hence, for $w \in W$ if $v = T^{-1}(w)$, then

$$\frac{|T^{-1}(w)|}{|w|} = \frac{|v|}{|Tv|} \le C$$

$\Rightarrow \|T^{-1}\| \le \frac{1}{c} < \infty$. The previous thm therefore implies T^{-1} is cts. □

Remark: $\frac{1}{\|T^{-1}\|}$ is called the conorm of T and represents the smallest factor by which T shrinks a vector

Cor: For finite dim'l normed spaces, all lin. trans. are cts. and isomorphisms are homeomorphisms. In particular, if a finite dim'l vector space has two norms, then the identity map yields a homeomorphism between the normed spaces. In particular, $T: \mathcal{M} \to \mathcal{L}$ is a homeomorphism.

Pf Exercise. □

5.2 Derivatives

Recall that for $U \subseteq \mathbb{R}$ open, $f: U \to \mathbb{R}$ a function, and $x \in U$, f has derivative $f'(x)$ at x if

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

We want to adapt this definition to functions of the form

$$f: U \to \mathbb{R}^m$$

where $U \subseteq \mathbb{R}^n$ is open. However, if $h \in \mathbb{R}^n$, we have no notion of division. & also $\frac{f(x+h)-f(x)}{|h|}$ doesn't work

Fortunately, we have the following alternative definition:

$$f(x+h) = f(x) + f'(x) \cdot h + R(h) \quad \text{where} \quad \lim_{h \to 0} \frac{R(h)}{|h|} = 0$$

Def: Let $U \subseteq \mathbb{R}^n$ be open and let $f: U \to \mathbb{R}^m$. We say f is differentiable at $p \in U$ with derivative $(Df)_p = T$ if $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ and if ~~defining~~

$$f(p+v) = f(p) + Tv + R(v) \implies \lim_{v \to 0} \frac{R(v)}{|v|} = 0$$

That is, the above defines a function R on nbhd of $0 \in \mathbb{R}^n$ and we require the limit of $\frac{R(v)}{|v|}$ be zero. R is called the Taylor remainder and is said to be sublinear if the limit $= 0$.

Remark: By necessity of matching "types" the derivative must be some kind of map from $\mathbb{R}^n$ to $\mathbb{R}^m$. We require it to be linear in keeping with the analogy that

$$f(p) + Tv$$

offers a good *linear* approximation of $f(p+v)$. Also note that when $n=m$, $T \in \mathcal{L}(\mathbb{R}, \mathbb{R})$ is just mult. by a scalar, so we recover

4/19/2017 our previous definition.

Ex: Consider $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(p_1, p_2) = (p_1 + p_2, p_2^2)$.
Claim that

$$(Df)_p = T = \begin{pmatrix} 1 & 1 \\ 0 & 2p_2 \end{pmatrix} \qquad p = (p_1, p_2) \in \mathbb{R}^2$$

Indeed, for $v = (v_1, v_2) \in \mathbb{R}$

$$R(v) = f(p+v) - f(p) - \begin{pmatrix} 1 & 1 \\ 0 & 2p_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$= \begin{pmatrix} p_1 + v_1 + p_2 + v_2 \\ (p_2 + v_2)^2 \end{pmatrix} - \begin{pmatrix} p_1 + p_2 \\ p_2^2 \end{pmatrix} - \begin{pmatrix} v_1 + v_2 \\ 2p_2 v_2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ v_2^2 \end{pmatrix}$$

and

$$\frac{|R(v)|}{|v|} = \frac{|(0, v_2^2)|}{\sqrt{v_1^2+v_2^2}} \leq \frac{|(0, v_2^2)|}{|v_2|} = |(0, |v_2|)| \to 0.$$

~~Note that as we vary p, $(Df)_p$ changes.~~

scr.

~~Def: If $U \subseteq \mathbb{R}^n$ is open and $f: U \to \mathbb{R}^m$ is differentiable at every $p \in U$, the function~~

~~$Df: U \to$~~

Remark: It is possible to visualize $(Df)_p$, though perhaps not as clearly as in 1-dim. See Figure 107 on page 283.

$$\text{circles} \xrightarrow{f} \text{wobbly ovals}$$
$$\text{circles} \xrightarrow{(Df)_p} \text{ellipses}$$
$$\text{lines} \xrightarrow{f} \text{curves}$$
$$\text{lines} \xrightarrow{(Df)_p} \text{lines}$$

Thm: If $f: U \to \mathbb{R}^m$ is diff'ble at $p \in U$, then ~~(Df)~~ it is uniquely determined by

$$(Df)_p(u) = \lim_{t \to 0} \frac{f(p+tu) - f(p)}{t} \quad \forall u \in \mathbb{R}^n$$

Pf: ~~Bef Def~~ For $u \in \mathbb{R}^n$ and $t \in \mathbb{R}$ we have by def.

$$\frac{f(p+tu) - f(p)}{t} = \frac{(Df)_p(tu) + R(tu)}{t}$$
$$= (Df)_p(u) + \frac{R(tu)}{|tu|} \cdot |u|$$

Since u is fixed and R is sublinear, we have
$$\lim_{t \to 0} \frac{f(p+tu) - f(p)}{t} = (Df)_p(u).$$

In particular, since limits are unique, $(Df)_p(u)$ is unique for each $u \in \mathbb{R}^n$ $\square$

~~Remark: The limit appearing in the previous theorem is sometimes called the directional derivative of f along u at p and is denoted $D_u f(p)$~~

Thm Differentiability implies continuity.

Pf: Suppose $f: U \to \mathbb{R}^m$ is diff'ble at p.

Then

$$|f(w) - f(p)| = |(Df)_p(w-p) + R(w-p)|$$
$$\leq \|(Df)_p\| \cdot |w-p| + |R(w-p)|$$

The latter two terms disappear as $w \to p$. $\blacksquare$

Def: Suppose $f: U \to \mathbb{R}^m$ is diff'ble at every $p \in U$. Then the function

$$Df: U \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$$
$$p \mapsto (Df)_p$$

is called the total derivative of f (or the Fréchet derivative)

Remark: Notation (a type-checking) is half the battle.

For $1 \leq i \leq m$, $f_i: U \to \mathbb{R}$ is ith component of f.

Def: Let $U \subseteq \mathbb{R}^n$ be open, $f: U \to \mathbb{R}^m$, and $1 \leq i \leq m$, $1 \leq j \leq m$. The ijth partial derivative of f at $p \in U$ is the limit, if it exists,

$$\frac{\partial f_i(p)}{\partial x_j} = \lim_{t \to 0} \frac{f_i(p + te_j) - f_i(p)}{t}$$

Corollary: If $f: U \to \mathbb{R}$ is differentiable at p, then all of its partial derivatives exist

and
$$(Df)_p = \begin{pmatrix} \frac{\partial f_1(p)}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n}(p) \\ \vdots & & \\ \frac{\partial f_m(p)}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial _n}(p) \end{pmatrix} = \left(\frac{\partial f_i}{\partial x_j}(p) \right|_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

Pf: By a previous theorem

j^{th} column $\longrightarrow (Df)_p(e_j) = \lim_{t \to 0} \frac{f(p+te_j) - f(p)}{t}$

$$= \left(\frac{\partial f_1(p)}{\partial x_j}, \cdots, \frac{\partial f_m(p)}{\partial x_j} \right) \qquad \square$$

Remark: The existence of the partial derivatives does not imply the existence of the total derivative. see the HW.

Thm If the partial derivatives of $f: U \to \mathbb{R}^m$ exist and are cts at $p \in U$, then f is diff'ble at p.

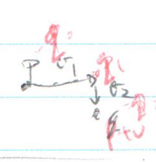
1/22/2018 Pf: Let $A \in M(m, n)$ be the matrix of partial derivatives at p:
$$[A]_{ij} = \frac{\partial f_i(p)}{\partial x_j} \qquad \begin{matrix} 1 \le i \le m \\ 1 \le j \le n \end{matrix}$$

Write $T = T_A$. We claim $(Df)_p = T$. To show this, we must prove
$$R(v) := f(p+v) - f(p) - T(v)$$
is sub-linear.

~~Denote $q = p+v$, and~~ let $\sigma = [\sigma_1, \sigma_2, \ldots, \sigma_n]$ be the path in $\mathbb{R}^n$ from p to ~~q~~ v which proceeds parallel to the std basis: if $v = (v_1, \ldots, v_n)$ define

$$\sigma_j(t) = q_{j-1} + t v_j e_j \qquad 0 \le t \le 1$$

where $q_{j-1} = p + \sum_{k<j} v_k e_k$, $q_0 = p$, $q_n = p + v$



$g(t) = f_i(p_1+v_1, \ldots, p_{j-1}+v_{j-1}, p_j + tv_j, p_{j+1}, \ldots, p_n)$

~~$g(t) = f_i(q_1, \ldots, q_{j-1}, p_j + t(q_j - p_j), p_{j+1}, \ldots, p_n)$~~

Observe that $g = f_i \circ \sigma_j(t) : [0,1] \to \mathbb{R}$ is diff'ble for each i,j. Thus the chain-rule and the MVT implies $\exists\, t_{ij} \in (0,1)$ s.t.

$$f_i(q_j) - f_i(q_{j-1}) = g(1) - g(0) = g'(t_{ij}) = \frac{\partial f_i(p_{ij})}{\partial x_j} \cdot v_j$$

where $p_{ij} = \sigma_j(t_{ij})$. Telescoping $f_i(p+v) - f_i(p) = f_i(q_n) - f_i(q_0)$ along σ yields:

$$\begin{aligned} R_i(v) &= f_i(p+v) - f_i(p) - (Tv)_i \\ &= \sum_{j=1}^{n} \left(f_i(q_j) - f_i(q_{j-1}) - \frac{\partial f_i(p)}{\partial x_j} v_j \right) \\ &= \sum_{j=1}^{n} \left(\frac{\partial f_i(p_{ij})}{\partial x_j} - \frac{\partial f_i(p)}{\partial x_j} \right) \cdot v_j \end{aligned}$$

Hence

$$\begin{aligned} \frac{|R(v)|}{|v|} &\le \sum_{i=1}^{m} \frac{|R_i(v)|}{|v|} \le \sum_{i=1}^{m} \sum_{j=1}^{n} \left| \frac{\partial f_i(p_{ij})}{\partial x_j} - \frac{\partial f_i(p)}{\partial x_j} \right| \cdot \frac{|v_j|}{|v|} \\ &\le \sum_{i=1}^{m} \sum_{j=1}^{n} \left| \frac{\partial f_i(p_{ij})}{\partial x_j} - \frac{\partial f_i(p)}{\partial x_j} \right| \end{aligned}$$

As $v \to 0$, $p_{ij} \to p$ and so cty of the partial derivatives implies $\frac{|R(v)|}{|v|} \to 0$. $\square$

Next we cover some basic rules of differentiation:

warning: Pugh tends to push technical details into the proof

Thm Let $f, g : \mathbb{R}^n \to \mathbb{R}^m$ be diff'ble. Then

(a) $D(f + cg) = Df + c\,Dg$, $c \in \mathbb{R}$

(b) $D(\text{const}) = 0$ and for $T \in \mathcal{L}$, $(DT)_p = T$ $\forall p \in \mathbb{R}^n$.

(c) $D(g \circ f) = Dg \circ Df$ (Chain Rule)

(d) $D(f \cdot g) = Df \cdot g + f \cdot Dg$ (product Rule) Leibniz rule

Pf (a) Exercise.

(b) If $f : \mathbb{R}^n \to \mathbb{R}^m$ constant, and $0 : \mathbb{R}^n \to \mathbb{R}^m$ is the zero map, then

$$R(v) = f(p+v) - f(p) - 0(v) = 0(v)$$

which is sublinear. Hence $(Df)_p = 0$.
For $T : \mathbb{R}^n \to \mathbb{R}^m$ a linear trans., we have

$$R(v) = T(p+v) - T(p) - T(v)$$
$$= T(p+v) - T(p+v) = 0$$

which is sublinear. Hence $(DT)_p = T$.

(c) ~~We~~ necessarily assume that
$$f : U \to \mathbb{R}^m$$
is diff'ble at some $p \in U$ and that
$$g : V \to \mathbb{R}^\ell$$
with $f(U) \subseteq V \subseteq \mathbb{R}^m$ and that g is diff'ble at $f(p) = q \in V$. We are claiming that
$$D(g \circ f)_p = \underbrace{(Dg)_q \circ (Df)_p}_{\text{comp as liner trans.}}$$

write $A = (Df)_p$ and $B = (Dg)_q$ and
$$f(p+v) = f(p) + Av + R_f(v)$$
$$g(q+w) = g(q) + Bw + R_g(w).$$

Define
$$e_f(v) = \begin{cases} \frac{|R_f(v)|}{|v|} & \text{if } v \neq 0 \\ 0 & \text{if } v = 0 \end{cases}$$

and similarly for $e_g(w)$. Then R_f being sublinear is equivalent to $\lim_{v \to 0} e_f(v) = 0$.

We compute
$$(g \circ f)(p+v) = g(f(p) + Av + R_f(v))$$
$$= g(q + Av + R_f(v))$$
$$= g(q) + B(Av + R_f(v)) + R_g(Av + R_f(v))$$
$$= g(q) + BAv + BR_f(v) + R_g(Av + R_f(v)).$$

~~Claim~~ Now
$$|BR_f(v)| \leq \|B\| \cdot |R_f(v)|$$
which is sublinear. ~~Also~~

$~~|Av + R_f(v)| \leq \|A\| \cdot |v| + \varepsilon_f(v) \cdot |v|~~$

So that

$$|R_g(Av + R_f(v))| \leq \varepsilon_g(Av + R_f(v)) \cdot |Av + R_f(v)|$$

$$\leq \varepsilon_g(Av + R_f(v)) (\|A\| + \varepsilon_f(v)) \cdot |v|$$

$$\frac{|R_g(Av + R_f(v))|}{|v|} \leq \varepsilon_g(Av + R_f(v)) (\|A\| + \varepsilon_f(v))$$

This sublinear since $v \to 0$ implies $Av + R_f(v) \to 0$.

We have shown

$$D(g \circ f)_p = BA = (Dg)_q \circ (Df)_p.$$

(d) Before finishing the proof, we make sense of the notation '$\circ$' which is a stand-in for any "bilinear map". ▨

Def: A map $\beta : V \times W \to Z$ is __bilinear__ if V, W, Z are vector spaces and for each fixed $v \in V$ and $w \in W$, the maps

$$\beta(v, \cdot) : W \to Z$$
$$\beta(\cdot, w) : V \to Z$$

are linear.

Ex (i) Multiplication in $\mathbb{R}$: $\quad \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$(x, y) \mapsto xy$

(ii) Inner product on $\mathbb{R}^n$: $\quad \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$(x, y) \mapsto \langle x, y \rangle$

(iii) Matrix mult: $\quad \mathcal{M}(m, k) \times \mathcal{M}(k, n) \to \mathcal{M}(m, n)$

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▨ Now, more precisely if $\beta : \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^m$ is bilinear, $f : U \to \mathbb{R}^k$ and $g : U \to \mathbb{R}^l$ are diff'ble at $p \in U$, then $\beta(f, g) : U \to \mathbb{R}^m$

$$x \mapsto \beta(f(x), g(x)) \in \mathbb{R}^m$$

is diff'ble at p with:

$$(D\,\beta(f, g))_p(v) = \beta((Df)_p(v), g(p)) + \beta(f(p), (Dg)_p(v))$$

To see this, we consider the following qty:

$$\|\beta\| := \sup\left\{ \frac{|\beta(v,w)|}{|v|\cdot|w|} : v, w \neq 0\right\}$$

which we claim is finite. To see this, observe that

$$\mathbb{R}^k \ni v \longmapsto \beta(v,\cdot) \in \mathcal{L}(\mathbb{R}^\ell, \mathbb{R}^m)$$

defines a linear trans.

$$T_\beta : \mathbb{R}^k \to \mathcal{L}(\mathbb{R}^\ell, \mathbb{R}^k), \quad T_\beta(v) = \beta(v,\cdot)$$

Since the vector spaces are finite, T_β is cts and therefore has finite operator norm. So for all $v \neq 0$, we have

$$\sup\left\{ \frac{\overbrace{|\beta(v,w)|}^{=|(T_\beta(v))(w)|}}{|v|\cdot|w|} : w \neq 0\right\} \leq \frac{\|T_\beta(v)\|}{|v|} \leq \|T_\beta\| < \infty.$$

Since this holds for all $v \neq 0$, we have $\|\beta\| \leq \|T_\beta\| < \infty$.

Now, let $A = (Df)_p$ and $B = (Dg)_p$ and write

$$R_f(v) = f(p+v) - f(p) - Av$$
$$R_g(v) = g(p+v) - g(p) - Bv$$

Then

$$\begin{aligned}\beta(f(p+v), g(p+v)) &= \beta(f(p) + Av + R_f(v), g(p) + Bv + R_g(v)) \\ &= \beta(f(p), g(p)) + \beta(Av, g(p)) + \beta(f(p), Bv) \\ &\quad + \beta(f(p), R_g(v)) + \beta(Av, Bv + R_g(v)) + \\ &\qquad \beta(R_f(v), g(p) + Bv + R_g(v))\end{aligned}$$

We'll show the last 3 terms ~~vanish~~ are sublinear.

Indeed:

$$|\beta(f(p), R_g(v))| \leq \|\beta\|\cdot|f(p)|\cdot|R_g(v)|$$
$$|\beta(Av, Bv + R_g(v))| \leq \|\beta\|\cdot\|A\|\cdot|v|\cdot|Bv + R_g(v)|$$
$$|\beta(R_f(v), g(p) + Bv + R_g(v))| \leq \|\beta\|\cdot|R_f(v)|\cdot|g(p) + Bv + R_g(v)|$$

Hence $\beta(f,g)$ is diff'ble at p w/

$$(D(\beta(f,g)))_p = \beta(A(\cdot), g(p)) + \beta(f(p), B(\cdot)) \quad \square$$

Thm A function $f: U \to \mathbb{R}^m$ is diff'ble at $p \in U$ iff each of its component functions f_i, $i=1,\dots,m$, is diff'ble at p. In this case, the derivative of its ith component is the ith row of its derivative.

Pf: Fix $1 \le i \le m$, and define $\pi_i : \mathbb{R}^m \to \mathbb{R}$ to be the projection $\pi_i(x_1,\dots,x_n) = x_i$. Then π_i is a lin. trans, so $(D\pi_i)_q = \pi_i$. ~~The chain rule~~ If f is diff'ble at p, then the chain rule implies

$$(Df_i)_p = (D(\pi_i \circ f))_p = (D\pi_i)_{f(p)} \circ (Df)_p$$
$$= \pi_i \circ (Df)_p.$$

Note ~~that the matrix~~ that $\pi_i = T_{[0,0,\dots,0,\underset{i\text{th comp.}}{1},0,\dots,0]}$

so that $\pi_i \circ (Df)_p$ corresponds to the ith row of $(Df)_p$.

Conversely, if each f_i, $i=1,\dots,m$, is diff'ble at p, define $T: \mathbb{R}^n \to \mathbb{R}^m$ by

$$T = ((Df_1)_p, \quad , (Df_m)_p)$$

We leave the rest as an exercise. $\square$

Remark: The previous theorem tells us that we do not lose any generality from assuming $m=1$. That is, multivariable calculus differs from one-variable calculus because of the multi-dimensionality of ~~the~~ domain, not ~~the~~ target.

Thm (Mean Value Theorem):
If $f: U \to \mathbb{R}^m$ is diff'ble on U and the segment $[p,q]$ (i.e. the line segment connecting points p and q) is contained in U, then

$$|f(q)-f(p)| \leq M|q-p|$$

where $M = \sup\{ \|(Df)_x\| : x \in [p,q]\}$

Pf: Fix any unit vector $u \in \mathbb{R}^n$. Consider

$$g : [0,1] \to \mathbb{R}$$

$$g(t) = \langle u, f(p + t(q-p))\rangle.$$

Then g is diff'ble and by a previous thm we know

$$g'(t) = \langle u, (Df)_{p+t(p-q)}(q-p)\rangle$$

In particular, the one-variable MVT implies $\exists\ \theta \in (0,1)$ s.t.

$$\langle u, f(q)-f(p)\rangle = g(1)-g(0) = g'(\theta)(1-0)$$

$$= \langle u, (Df)_{p+\theta(p-q)}(q-p)\rangle.$$

$$\leq M|q-p|.$$

Taking $u = \dfrac{f(q)-f(p)}{|f(q)-f(p)|}$ yields the result. $\square$

Remark: We can state a more direct multivariable analogue of the 1-D MVT:

$$f(p)-f(q) = (Df)_\theta(p-q)$$

for some $\theta \in [p,q]$. But this doesn't hold in general. ~~(See HW [pg 361, #17]. However, with some additional assumptions of f, we can upgrade the previous theorem.~~ Consider $p=\pi$, $q=2\pi$, $f : \mathbb{R} \to \mathbb{R}^2$, $f(t) = (\cos(t), \sin(t))$.

Def: We say $f : U \to \mathbb{R}^m$ is of class C^1 (on U) if it is differentiable on U and the map

$$U \longrightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$$

$$p \longmapsto (Df)_p$$

is cts. (here $\mathcal{L}(\mathbb{R}^n,\mathbb{R}^m)$ is a normed space with norm given by the operator norm).

¿Exercise? Show the op norm is a norm.

Thm (C^1 Mean Value Theorem):

If $f: U \to \mathbb{R}^m$ is of class C^1 and the segment $[p,q]$ is in U then

$$(*) \quad f(p) - f(q) = T(p-q)$$

where

$$(**) \quad T = \int_0^1 (Df)_{p+t(q-p)}\, dt$$

is the average derivative of f on the segment.

Conversely, ~~[illegible]~~ $U \ni (p,q) \mapsto T_{pq}$ cts ~~[illegible] that $p \in U$ [illegible] $p \in B \subset U$ an open [illegible] ball centered at p. If there exists a family of linear trans $\{T_q\}_{q \in B} \subset \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ s.t. $B \ni q \mapsto T_q$~~ is cts, and for which $(*)$ holds, then f is of class C^1 (on B) and $(Df)_p = T_p$.

Remark: The integral in $(**)$ is defined by a limit of Riemann sums in the same way we define the Riemann integral. The fact that $p \mapsto (Df)_p$ is cts implies this limit always exists and defines a lin. trans. Alternatively, integrate matrix entries.

Pf (C^1 MVT): Fix $i \in \{1, \dots, m\}$ and consider ~~applying the FTC and the real MVT to~~

$$g_i : [0,1] \to \mathbb{R}$$
$$g(t) := f_i \circ \sigma(t)$$

where $\sigma(t) = p + t(q-p)$. The FTC and the MVT imply:

$$\begin{aligned} f_i(q) - f_i(p) = g(1) - g(0) &= \int_0^1 g'(t)\,dt \\ &= \int_0^1 \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}(\sigma(t))(q_j - p_j)\,dt \\ &= \sum_{j=1}^n \int_0^1 \frac{\partial f_i}{\partial x_j}(\sigma(t))\,dt\,(q_j - p_j) \\ &= [T(q-p)]_i \end{aligned}$$

To check the converse, write $q = p + v$ for $q \in B$ (so v is near 0). The Taylor remainder for f at p is:

$$R(v) = f(p+v) - f(p) - T_{pp}(v) = T_{pq}(p+v-p) - T_{pp}(v)$$

$$= (T_{pq} - T_{pp})(v).$$

Since $T_{p,p+v} \to T_{pp}$ as $v \to 0$, the above is sublinear. Thus $T_{pp} = (Df)_p$. Since $p \in U$ was arbitrary, we have $T_{q,q} = (Df)_q$ $\forall q \in U$ and moreover, $q \mapsto T_{q,q}$ is cts. Thus f is of class C^1. $\square$

Cor: Assume that $U \subseteq \mathbb{R}^n$ is connected. If $f : U \to \mathbb{R}^m$ is diff'ble with $(Df)_p = 0$ $\forall p \in U$, then f is constant.

Pf Exercise. $\square$

We now examine a case when it is permitted to differentiate past the integral sign:

Thm: Assume $f : [a,b] \times (c,d) \to \mathbb{R}$ is cts and that $\frac{\partial f(x,y)}{\partial y}$ exists and is cts. Then

$$F(y) := \int_a^b f(x,y)\,dx$$

is of class C^1 and

$$F'(y) = \int_a^b \frac{\partial f(x,y)}{\partial y}\,dx.$$

Pf: By the C^1 MVT, if h is suff'ly small then

$$\frac{F(y+h) - F(y)}{h} = \frac{1}{h} \int_a^b \left(\int_0^1 \frac{\partial f(x, y+th)}{\partial y}\,dt \right) h\,dx$$

By cty of $\frac{\partial f}{\partial y}$,

(avg.) $$\int_0^1 \frac{\partial f(x, y+th)}{\partial y}\,dt \xrightarrow{h \to 0} \frac{\partial f(x,y)}{\partial y}$$

Thus $$F'(y) = \int_a^b \frac{\partial f(x,y)}{\partial y}\,dx.$$

That F' is cts (so F is C^1), follows from cty of $\frac{\partial f}{\partial y}$. $\square$