The Weierstrass Function

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We let $\sin, \cos\colon \mathbb{R} \to \mathbb{R}$ be defined in the usual geometric way, extended to all of $\mathbb{R}$. We will assume the following facts about these functions:

(a) $\sin$ and $\cos$ are continuous on $\mathbb{R}$;

(b) $|\sin(x)|, |\cos(x)| \leq 1$ for all $x \in \mathbb{R}$;

(c) $\left|\frac{\sin(x)}{x}\right| \leq 1$ for all $x \in \mathbb{R} \setminus \{0\}$;

(d) $\cos(x) - \cos(y) = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$ for all $x, y \in \mathbb{R}$;

(e) $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ for all $x, y \in \mathbb{R}$.

These can be derived by considering, for example, the power series representations:

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad \text{and} \qquad \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

The main goal of these notes is to prove the following theorem:

Theorem (Karl Weierstrass, 1872)**.** *Let $a \in (0,1)$ and let b be an odd integer such that $ab > 1 + \frac{3\pi}{2}$. Then the series*

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$

converges uniformly on $\mathbb{R}$ and defines a continuous but nowhere differentiable function.

The function appearing in the above theorem is called the Weierstrass function. Before we prove the theorem, we require the following lemma:

Lemma (The Weierstrass M-test)**.** *Let (E,d) be a metric space, and for each $n \in \mathbb{N}$ let $f_n\colon E \to \mathbb{R}$ be a function. Suppose that for each $n \in \mathbb{N}$, there exists $M_n > 0$ such that*

$$|f(x)| \leq M_n \qquad \forall x \in E.$$

If the series $\sum_{n=1}^{\infty} M_n$ converges, then the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on E

Proof. Let $\epsilon > 0$. The Cauchy criterion for the convergence of a series implies there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$ with $n < m$ we have

$$|M_{n+1} + M_{n+2} + \cdots + M_m| = M_{n+1} + M_{n+2} + \cdots + M_m < \epsilon.$$

Consequently, for all $n, m \geq N$ with $n < m$ we have for all $x \in E$

$$\left|\sum_{i=1}^{m} f_i(x) - \sum_{i=1}^{n} f_i(x)\right| = |f_{n+1}(x) + \cdots + f_m(x)| \leq |f_{n+1}(x)| + \cdots + |f_m(x)| \leq M_{n+1} + \cdots + M_m < \epsilon.$$

That is, the sequence of partial sums $\left(\sum_{i=1}^{n} f_i\right)_{n\in\mathbb{N}}$ satisfies the Cauchy criterion for functions. So by a proposition from lecture we know that these partial sums converge uniformly to the series $\sum_{n=1}^{\infty} f_n$. $\square$

Proof of Theorem. Since $|a^n \cos(b^n \pi x)| \leq a^n$ for all $x \in \mathbb{R}$ and $\sum_{n=0}^{\infty} a^n$ converges, the series converges uniformly by the Weierstrass M-test. Moreover, since the partial sums are continuous (as finite sums of continuous functions), their uniform limit f is also continuous.

To see that f is nowhere differentiable, we will show for each $x_0 \in \mathbb{R}$ that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

does not exist. In particular, we'll show that as x approaches x_0 from above and below, the respective difference quotients oscillate wildly between larger and larger positive and negative values.

Fix $x_0 \in \mathbb{R}$. For each $m \in \mathbb{N}$, let $\alpha_m \in \mathbb{Z}$ be such that

$$b^m x_0 - \alpha_m \in \left(-\frac{1}{2}, \frac{1}{2}\right].$$

Define

$$x_m := b^m x_0 - \alpha_m \qquad y_m := \frac{\alpha_m - 1}{b^m} \qquad z_m := \frac{\alpha_m + 1}{b^m}.$$

Observe that

$$y_m - x_0 = -\frac{1 + x_m}{b^m} < 0 < \frac{1 - x_m}{b^m} = z_m - x_0.$$

Thus $y_m < x_0 < z_m$,

$$\lim_{m \to \infty} |y_m - x_0| = \lim_{m \to \infty} x_0 - y_m = \lim_{m \to \infty} \frac{1 + x_m}{b^m} = 0,$$

and

$$\lim_{m \to \infty} |z_m - x_0| = \lim_{m \to \infty} z_m - x_0 = \lim_{m \to \infty} \frac{1 - x_m}{b^m} = 0.$$

That is, $(y_m)_{m \in \mathbb{N}}$ and $(z_m)_{m \in \mathbb{N}}$ are (meticulously constructed) sequences converging to x_0, but from above and below x_0, respectively. We will examine the difference quotients for f proceeding along $x = y_m$, $m \in \mathbb{N}$, and $x = z_m$, $m \in \mathbb{N}$. First,

$$\begin{aligned} \frac{f(y_m) - f(x_0)}{y_m - x_0} &= \frac{\sum_{n=0}^{\infty} a^n \cos(b^n \pi y_m) - \sum_{n=0}^{\infty} a^n \cos(b^n \pi x_0)}{y_m - x_0} \\ &= \sum_{n=0}^{\infty} a^n \frac{\cos(b^n \pi y_m) - \cos(b^n \pi x_0)}{y_m - x_0} \\ &= \sum_{n=0}^{m-1} (ab)^n \frac{\cos(b^n \pi y_m) - \cos(b^n \pi x_0)}{b^n (y_m - x_0)} + \sum_{n=0}^{\infty} a^{n+m} \frac{\cos(b^{n+m} \pi y_m) - \cos(b^{n+m} \pi x_0)}{y_m - x_0}. \end{aligned}$$

We denote the two sums in the last expression by S_1 and S_2, respectively. Roughly speaking, we will show that S_1 is small while S_2 is big. Using property (d), we have

$$\begin{aligned} S_1 &= \sum_{n=0}^{m-1} (ab)^n \frac{-2}{b^n (y_m - x_0)} \sin\left(\frac{b^n \pi (y_m + x_0)}{2}\right) \sin\left(\frac{b^n \pi (y_m - x_0)}{2}\right) \\ &= \sum_{n=0}^{m-1} -\pi (ab)^n \sin\left(\frac{b^n \pi (y_m + x_0)}{2}\right) \frac{\sin\left(\frac{b^n \pi (y_m - x_0)}{2}\right)}{\frac{\pi b^n (y_m - x_0)}{2}}. \end{aligned}$$

Using the triangle inequality and properties (b) and (c) we have

$$|S_1| \leq \sum_{n=0}^{m-1} \pi (ab)^n 1 \cdot 1 = \pi \frac{(ab)^m - 1}{ab - 1} < \pi \frac{(ab)^m}{ab - 1}.$$

Thus, there exists $\epsilon_1 \in (-1, 1)$ such that $S_1 = \epsilon_1 \frac{\pi (ab)^m}{ab - 1}$.

Next, we handle S_2. First, recall that $y_m = \frac{\alpha_m - 1}{b^m}$, that α_m is an integer, and that b is an odd integer. Thus

$$\cos\left(b^{n+m}\pi y_m\right) = \cos\left(b^n \pi(\alpha_m - 1)\right) = (-1)^{b^n(\alpha_m - 1)} = (-1)^{\alpha_m - 1} = -(-1)^{\alpha_m}.$$

Also, recall that $x_m = b^m x_0 - \alpha_m$ so that using property (e) we have

$$\begin{aligned} \cos\left(b^{n+m}\pi x_0\right) &= \cos\left(b^n\pi(x_m + \alpha_m)\right) \\ &= \cos(b^n\pi x_m)\cos(b^n\pi\alpha_m) - \sin(b^n\pi x_m)\sin(b^n\pi\alpha_m) \\ &= (-1)^{b^n\alpha_m}\cos(b^n\pi x_m) - 0 \\ &= (-1)^{\alpha_m}\cos(b^n\pi x_m). \end{aligned}$$

Using these computations, we have

$$\begin{aligned} S_2 &= \sum_{n=0}^{\infty} a^{n+m}\frac{-(-1)^{\alpha_m} - (-1)^{\alpha_m}\cos(b^n\pi x_m)}{y_m - x_0} \\ &= \sum_{n=0}^{\infty} a^{n+m}(-1)(-1)^{\alpha_m}\frac{1 + \cos(b^n\pi x_m)}{-\frac{1+x_m}{b^m}} \\ &= (ab)^m(-1)^{\alpha_m}\sum_{n=0}^{\infty} a^n\frac{1 + \cos(b^n\pi x_m)}{1 + x_m}. \end{aligned}$$

Recall that $x_m \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ so the terms in the sum in the last expression are non-negative. Consequently,

$$\sum_{n=0}^{\infty} a^n\frac{1 + \cos(b^n\pi x_m)}{1 + x_m} \geq \frac{1 + \cos(\pi x_m)}{1 + x_m} \geq \frac{1}{1 + \frac{1}{2}} = \frac{2}{3}.$$

So there exists $\eta_1 \geq 1$ such that $S_2 = (ab)^m(-1)^{\alpha_m}\eta_1\frac{2}{3}$.

Putting our computations for S_1 and S_2 together yields

$$\begin{aligned} \frac{f(y_m) - f(x_0)}{y_m - x_0} = S_1 + S_2 &= \epsilon_1\frac{\pi(ab)^m}{ab - 1} + (ab)^m(-1)^{\alpha_m}\eta_1\frac{2}{3} \\ &= (-1)^{\alpha_m}(ab)^m\eta_1\left(\frac{2}{3} + (-1)^{\alpha_m}\frac{\epsilon_1}{\eta_1}\frac{\pi}{ab - 1}\right) \end{aligned}$$

Recall our assumption that $ab > 1 + \frac{3\pi}{2}$, which is equivalent to $\frac{\pi}{ab-1} < \frac{2}{3}$. Using $|\epsilon_1| < 1$ and $\eta \geq 1$, we have

$$\frac{2}{3} + (-1)^{\alpha_m}\frac{\epsilon_1}{\eta_1}\frac{\pi}{ab - 1} > \frac{2}{3} - \frac{\pi}{ab - 1} > 0.$$

Consequently, the sign of $\frac{f(y_m) - f(x_0)}{y_m - x_0}$ is completely determined by $(-1)^{\alpha_m}$ and

$$\left|\frac{f(y_m) - f(x_0)}{y_m - x_0}\right| > (ab)^m\left(\frac{2}{3} - \frac{\pi}{ab - 1}\right)$$

Thus, not only does the difference quotient alternate signs rapidly, but its magnitude tends to $+\infty$ as $m \to \infty$. Since $\lim_{m\to\infty} y_m = x_0$, this is enough to show that $\lim_{x\to x_0}\frac{f(y_m) - f(x_0)}{y_m - x_0}$ does not exist. We will show something slightly stronger: the same behavior also occurs along $(z_m)_{m\in\mathbb{N}}$.

Using the same breakdown as before, we can write

$$\frac{f(z_m) - f(x_0)}{z_m - x_0} = S_1' + S_2',$$

and the same argument yields $S_1' = \epsilon_2\frac{\pi(ab)^m}{ab-1}$ for some $\epsilon_2 \in (-1, 1)$. Using $z_m - x_0 = \frac{1-x_m}{b^m}$ we have

$$\begin{aligned} S_2' &= \sum_{n=0}^{\infty} a^{n+m}\frac{-(-1)^{\alpha_m} - (-1)^{\alpha_m}\cos(b^n\pi x_m)}{\frac{1-x_m}{b^m}} \\ &= -(ab)^m(-1)^{\alpha_m}\sum_{n=0}^{\infty} a^n\frac{1 + \cos(b^n\pi x_m)}{1 - x_m}. \end{aligned}$$

Since $x_m \in \left(-\frac{1}{2}, \frac{1}{2}\right]$, the terms in the sum in the last expression are non-negative. Consequently,

$$\sum_{n=0}^{\infty} a^n \frac{1+\cos(b^n \pi x_m)}{1-x_m} \geq \frac{1+\cos(\pi x_m)}{1-x_m} > \frac{1}{1-(-\frac{1}{2})} = \frac{2}{3}.$$

So there exists $\eta_2 \geq 1$ such that $S_2' = -(ab)^m(-1)^{\alpha_m}\eta_2 \frac{2}{3}$. Then

$$\begin{aligned} \frac{f(z_m)-f(x_0)}{z_m - x_0} = S_1' + S_2' &= \epsilon_2 \frac{\pi (ab)^m}{ab-1} - (-1)^{\alpha_m}(ab)^m \eta_2 \frac{2}{3} \\ &= -(-1)^{\alpha_m}(ab)^m \eta_2 \left(\frac{2}{3} - (-1)^{\alpha_m}\frac{\epsilon_2}{\eta_2}\frac{\pi}{ab-1}\right). \end{aligned}$$

Just as before we have

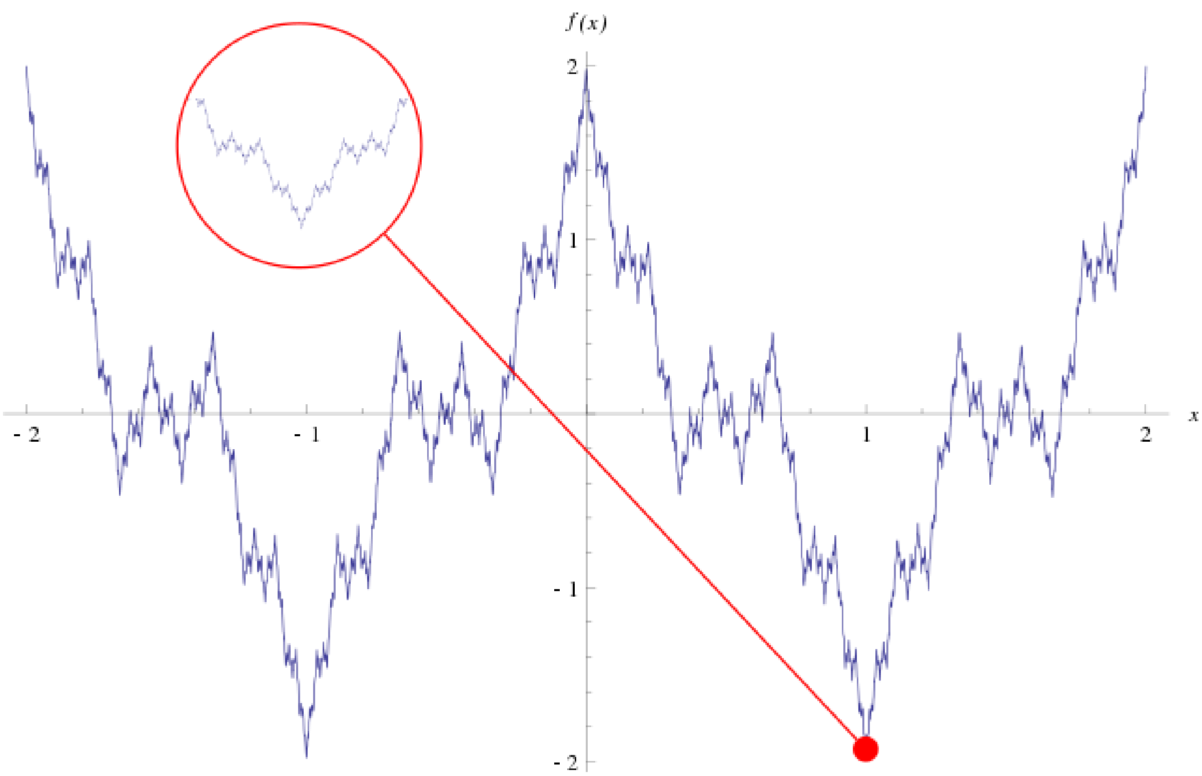
$$\frac{2}{3} - (-1)^{\alpha_m}\frac{\epsilon_2}{\eta_2}\frac{\pi}{ab-1} > \frac{2}{3} - \frac{\pi}{ab-1} > 0,$$

so that the sign of $\frac{f(z_m)-f(x_0)}{z_m-x_0}$ has sign completely determined by $-(-1)^{\alpha_m}$. Also,

$$\left|\frac{f(z_m)-f(x_0)}{z_m-x_0}\right| > (ab)^m\left(\frac{2}{3} - \frac{\pi}{ab-1}\right) \overset{m\to\infty}{\longrightarrow} +\infty$$

So the same behavior occurs to the right of x_0. □

The graph of the Weierstrass function



The rough shape of the graph is determined by the $n = 0$ term in the series: $\cos(\pi x)$. The higher-order terms create the smaller oscillations. With b carefully chosen as in the theorem, the graph becomes so jagged that there is no reasonable choice for a tangent line at any point; that is, the function is nowhere differentiable.