12/4/2017

Bonus Lecture: Infinite Series VII.2

Given a sequence $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$, we can form a new sequence using addition:

$$a_1,\ a_1+a_2,\ a_1+a_2+a_3,\ \ldots,\ a_1+a_2+\cdots+a_N = \sum_{n=1}^{N} a_n,$$

<u>Def</u>: For $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$, the quantity $\sum_{n=1}^{N} a_n$ is called a partial sum, and the sequence formed by these partial sums is called the (infinite) series and is denoted

$$\sum_{n=1}^{\infty} a_n$$

We say the infinite series converges to $A \in \mathbb{R}$ if

$$\lim_{N\to\infty} \sum_{n=1}^{N} a_n = A$$

We ~~abuse~~ abuse notation and denote: $A = \sum_{n=1}^{\infty} a_n$. and ~~[illegible]~~ call A the sum of the series

<u>Ex</u>: On homework, showed $\sum_{n=0}^{\infty} x^n$ converges iff $|x|<1$. Used formula for partial sums:

$$\sum_{n=0}^{N} x^n = \frac{1-x^{N+1}}{1-x}.$$

Really, the question was about sequences of functions.

<u>Def</u> Given a sequence of functions $f_n: E \to \mathbb{R}$, $n\in\mathbb{N}$, defined on a metric space (E,d), we say the series $\sum_{n=1}^{\infty} f_n$ converges at $x\in E$ if the series $\sum_{n=1}^{\infty} f_n(x)$ converges.

We say the series $\sum_{n=1}^{\infty} f_n$ converges pointwise on E if $\sum_{n=1}^{\infty} f_n(x)$ converges ~~pointwise~~ for all $x \in E$

(this is the same as saying the sequence $(f_1+f_2+\cdots+f_N)_{N\in\mathbb{N}}$ conv. ptwise on E.)

cleary

If the seq. of functions $(f_1 + f_2 + \cdots + f_N)_{N \in \mathbb{N}}$ converges uniformly on E, we say the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on E.

clear

Ex: On Homework, showed $\sum_{n=1}^{\infty} x^n$ converges ~~uniformly on~~ ptwise on $(-1,1)$, and uniformly on $[-r,r]$ for any $0<r<1$.

Prop: For $(a_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}$, the series $\sum_{n=1}^{\infty} a_n$ converges iff $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $\forall n, m \geq N$, with $n < m$

$$|a_{n+1} + a_{n+2} + \cdots + a_m| < \varepsilon$$

Pf: $$\left| \sum_{i=1}^{m} a_i - \sum_{i=1}^{n} a_i \right|$$ and $\mathbb{R}$ is complete

so convergence is ~~equivalent~~ equivalent to being Cauchy □

Cor: If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$

Ex ① It follows easily from this that $\sum_{n=1}^{\infty} x^n$ does not converge if $|x| \geq 1$.

Ex ② (Harmonic Series) $\sum_{n=1}^{\infty} \frac{1}{n}$ does **not** converge, even though $\lim_{n\to\infty} \frac{1}{n} = 0$.

Set $\varepsilon = \frac{1}{2}$ and let $N \in \mathbb{N}$. Take $n = N$, $m = 2N$.

Then

$$\left|\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{m}\right| = \frac{1}{n+1} + \cdots + \frac{1}{m} \geq \underbrace{\frac{1}{m} + \frac{1}{m} + \cdots + \frac{1}{m}}_{N}$$

$$= N \cdot \frac{1}{m} = \frac{N}{2N} = \frac{1}{2} \geq \varepsilon.$$

Def: We say $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges.

Observe that

$$|a_{n+1} + \cdots + a_m| \leq |a_{n+1}| + \cdots + |a_m| = \big|\, |a_{n+1}| + \cdots + |a_m| \,\big|$$

implies ~~convergence~~ absolute convergence $\Rightarrow$ convergence.

(of $(a_n)_{n \in \mathbb{N}}$)

Cor (Comparison test)
Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series such that $|a_n| \leq b_n$ for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

~~Prop (Ratio Test)~~
~~If $\sum_{n=1}^{\infty} a_n$ is an infinite series of non-zero real numbers and~~

Prop (Ratio Test)
Let $(a_n)_{n \in \mathbb{N}} \subseteq \mathbb{R} \setminus \{0\}$.

$$\text{If } \begin{cases} \limsup_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| < 1 \\ \liminf_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| > 1 \end{cases} \text{ then } \sum_{n=1}^{\infty} a_n \begin{cases} \text{converges absolutely} \\ \text{diverges.} \end{cases}$$

Pf: Let $r < 1$ satisfy

$$\limsup_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| \leq r$$

Then $\exists N \in \mathbb{N}$ s.t. $\sup\{\left|\frac{a_{n+1}}{a_n}\right| : n \geq N\} \leq r$
Consequently, $\forall n \geq N$ we have

$$|a_n| \leq r \cdot |a_{n-1}| \leq r^2 |a_{n-2}| \leq \cdots \leq r^{n-N} |a_N|.$$

Since $r < 1$,

$$\sum_{n=N}^{\infty} r^{n-N} |a_N|$$

converges. By the comparison test, $\sum_{n=N}^{\infty} a_n$ converges absolutely $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges absolutely.

In the other case, let $r \geq 1$ be such that

$$\liminf_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| > r.$$

Then $\exists N \in \mathbb{N}$ s.t. $\inf\{\left|\frac{a_{n+1}}{a_n}\right| : n \geq N\} \geq r$.
Hence, $\forall n \geq N$

$$|a_n| \geq r \cdot |a_{n-1}| \geq \cdots \geq r^{n-N} |a_N| \geq |a_N|$$

Thus $\lim_{n\to\infty} |a_n| \neq 0$ and so series does not converge. $\square$

Function def ⟸ 0

<u>Prop</u> Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions $f_n : E \to \mathbb{R}$. Then $\sum_{n=1}^{\infty} f_n$ converges unif. iff $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ s.t. $\forall n, m \geq N$ with $n < m$

$$\sup\{|f_{n+1}(x) + \cdots + f_m(x)| : x \in E\} \leq \varepsilon$$

<u>Pf</u>: This is simply our previous Cauchy criterion. $\square$

<u>Thm</u> (Weierstrass M-Test)
Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions $f_n : E \to \mathbb{R}$. Suppose $\exists M_n > 0$ s.t. $|f_n(x)| \leq M_n\ \forall x \in E$. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly on E.

<u>Pf</u>: Let $\varepsilon > 0$. Since $\sum_{n=1}^{\infty} M_n$ converges, $\exists N \in \mathbb{N}$ s.t. $\forall n, m \geq N$ with $n < m$

$$|M_{n+1} + \cdots + M_m| = M_{n+1} + \cdots + M_m < \varepsilon.$$

Consequently $\forall x \in E$

$$|f_{n+1}(x) + \cdots + f_m(x)| \leq |f_{n+1}(x)| + \cdots + |f_m(x)|$$
$$\leq M_{n+1} + \cdots + M_m < \varepsilon.$$

By the previous prop, $\sum_{n=1}^{\infty} f_n$ conv. unif. $\square$

<u>Ex</u> (The Weierstrass Function)
Let $a \in (0,1)$ and let b be an odd integer such that $a \cdot b > 1 + \frac{3\pi}{2}$ (e.g. $b \geq 7$). Then

$$\sum_{n=0}^{\infty} a^n \cos(b^n \cdot \pi \cdot x)$$

converges $\forall x \in \mathbb{R}$ and defines a continuous but <u>nowhere</u> differentiable function.

That this ~~function~~ function exists and is cts follows from the Weierstrass M-test and the fact that uniform limits of cts functions are cts. Showing that it is nowhere differentiable is considerably more involved...

Other Pathological Examples:

Suppose $\sum_{n=1}^{\infty} a_n$ converges, but does not converge absolutely. Then for any real number $x \in \mathbb{R}$, $\exists$ a bijection $\sigma: \mathbb{N} \to \mathbb{N}$ s.t.

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = x.$$