$$\int_a^b f_2(x) - f_1(x)\,dx = \sum_{i=1}^{N} (f(x_i'') - f(x_i'))\cdot(x_i - x_{i-1})$$
$$< \sum_{i=1}^{N} \frac{\varepsilon}{b-a}\cdot(x_i - x_{i-1}) = \varepsilon.$$

Thus the prop. $\Rightarrow$ f is int'ble. $\square$

The Fundamental Theorem of Calculus VI.4

<u>Prop</u> Let $a, b, c \in \mathbb{R}$, $a < b < c$, and let $f: [a,c] \to \mathbb{R}$. Then f is int'ble on $[a,c]$ if and only if f is int'ble on $[a,b]$ and $[b,c]$, in which case:

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

<u>Pf</u> ($\Longrightarrow$): Suppose f is int'ble on $[a,c]$. Let $\varepsilon > 0$ and find step functions $f_1, f_2 : [a,c] \to \mathbb{R}$ s.t.

$$f_1(x) \le f(x) \le f_2(x) \qquad \forall x \in [a,c]$$

and s.t.

$$\int_a^c f_2(x) - f_1(x)\,dx < \varepsilon.$$

Then restricting these functions to $[a,b]$ and $[b,c]$ (and using the formula for step functions which is easily checked), we see that f is integrable on $[a,b]$ and $[b,c]$. Using the formula for step functions, we can see that it holds for f as well.

Skip

($\Longleftarrow$): Suppose f is int'ble on $[a,b]$ and $[b,c]$. Let $\varepsilon > 0$. Then $\exists\, h_1, h_2 : [a,b] \to \mathbb{R}$, $k_1, k_2 : [b,c] \to \mathbb{R}$ s.t.

$$h_1(x) \le f(x) \le h_2(x) \qquad \forall x \in [a,b]$$
$$k_1(x) \le f(x) \le k_2(x) \qquad \forall x \in [b,c]$$

and s.t.

$$\int_a^b h_2(x) - h_1(x)\,dx < \varepsilon/2$$
$$\int_b^c k_2(x) - k_1(x)\,dx < \varepsilon/2$$

Skip

Define, for $j=1,2$

$$f_j(x) = \begin{cases} h_j(x) & \text{if } a \le x \le b \\ k_j(x) & \text{if } b < x \le c \end{cases}$$

then $f_1, f_2 : [a,b] \to \mathbb{R}$ are step functions s.t.

$$f_1(x) \le f(x) \le f_2(x) \qquad \forall x \in [a,b]$$

and s.t.

$$\int_a^c f_2(x) - f_1(x)\,dx = \int_a^b f_2(x) - f_1(x)\,dx + \int_b^c f_2(x) - f_1(x)\,dx$$
$$= \int_a^b h_2(x) - h_1(x)\,dx + \int_b^c k_2(x) - k_1(x)\,dx$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad \square$$

<u>Def</u>: If $f:[a,b] \to \mathbb{R}$ is int'ble, we set

$$\int_b^a f(x)\,dx := -\int_a^b f(x)\,dx$$

<u>Cor</u>: For $I \subseteq \mathbb{R}$ an interval, let $f: I \to \mathbb{R}$ and let $a, b, c \in I$. If two of

$$\int_a^b f(x)dx, \quad \int_b^c f(x)dx, \quad \int_a^c f(x)\,dx$$

exist, then the third does and

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$$

Skip

<u>Pf</u>: If $a < b < c$, this is simply the previous prop. In any other case we just apply the above definition. $\square$

<u>Cor</u> Let $f:[a,b] \to \mathbb{R}$ be int'ble. Then for any $c, d \in [a,b]$, $\int_c^d f(x)dx$ exists. Moreover, if $|f(x)| \le M$ $\forall x$ between c and d, then

$$\left|\int_c^d f(x)\,dx\right| \le M|c-d|.$$

<u>Thm (the Fundamental Theorem of Calculus)</u>
Let $U \subseteq \mathbb{R}$ be an open interval, and let $f: U \to \mathbb{R}$ be a cts function.
Fix $a \in U$ and define $F: U \to \mathbb{R}$ by:

$$F(x) = \int_a^x f(t)\,dt$$

Then F is diff'ble with $F'(x) = f(x)$

<u>Pf</u>: Note that f cts $\Rightarrow$ $F(x)$ exists for each $x \in U$. We need to show:

$$\lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0) \quad \forall x_0 \in U$$

Fix $x_0 \in U$ and let $\varepsilon > 0$. For $x \in U \setminus \{x_0\}$ we have:

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| = \frac{1}{|x - x_0|} \left| \int_a^x f(t)\,dt - \int_a^{x_0} f(t)\,dt - \int_{x_0}^x f(x_0)\,dt \right|$$

$$= \frac{1}{|x - x_0|} \left| \int_{x_0}^x f(t) - f(x_0)\,dt \right|$$

Now, let $\delta > 0$ be s.t. $|f(t) - f(x_0)| < \varepsilon$ whenever $|t - x_0| < \delta$. Observe that if $|x - x_0| < \delta$, then $|t - x_0| < \delta$ for any t between x and x_0. Thus if $|x - x_0| < \delta$, we can continue our above estimate using the previous corollary:

$$\leq \frac{1}{|x - x_0|} \cdot \varepsilon |x - x_0| = \varepsilon$$

Thus F is diff'ble with $F' = f$. $\square$

<u>Def</u> For a function $f: [a,b] \to \mathbb{R}$, an <u>antiderivative</u> of f is a diff'ble function $F: [a,b] \to \mathbb{R}$ s.t. $F' = f$.

<u>Cor 1</u>: Every cts $f: [a,b] \to \mathbb{R}$ has an anti-derivative.

<u>Cor 2</u>: For $U \subseteq \mathbb{R}$ an open interval and $F: U \to \mathbb{R}$ diff'ble with $F' = f$ cts. Then $\forall a,b \in U$

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

<u>Pf</u>: Let $g(x) = \int_a^x f(t)\,dt - F(x)$. Then $g'(x) = 0$ $\forall x \in U$ by the FTC. Thus $g(x) = C$ for some constant $C \in \mathbb{R}$. In particular:

$$0 = g(b) - g(a) = \left(\int_a^b f(t)\,dt - F(b)\right) - \left(\int_a^a f(t)\,dt - F(a)\right)$$
$$= \int_a^b f(t)\,dt - (F(b) - F(a)). \quad \square$$

<u>Cor 3 (Change of Variables Formula)</u>
Let $U, V \subseteq \mathbb{R}$ be open intervals. Let $\varphi: U \to V$ be diff'ble with $\varphi': U \to \mathbb{R}$ cts. Let $f: V \to \mathbb{R}$ be cts. Then $\forall a,b \in U$

$$\int_{\varphi(a)}^{\varphi(b)} f(v)\,dv = \int_a^b f(\varphi(u)) \cdot \varphi'(u)\,du$$

<u>Pf</u>: Define $F: V \to \mathbb{R}$ by

$$F(y) := \int_{\varphi(a)}^{y} f(v)\,dv$$

Then F is diff'ble with $F' = f$. Consider $G: U \to \mathbb{R}$ defined by

$$G(x) = \int_{\varphi(a)}^{\varphi(x)} f(v)\,dv = F \circ \varphi(x).$$

By the chain-rule, G is diff'ble with

$$G'(x) = F'(\varphi(x)) \cdot \varphi'(x) = f(\varphi(x)) \cdot \varphi'(x)$$

Thus the FTC implies

$$\cancel{G(b) = G(b) - G(a) = \int_a^b}$$

$$\int_{\varphi(a)}^{\varphi(b)} f(v)\,dv = G(b) = G(b) - G(a) = \int_a^b G'(u)\,du = \int_a^b f(\varphi(u)) \cdot \varphi'(u)\,du. \quad \square$$