Cor 2: If $f, g : (a,b) \to \mathbb{R}$ are diff'ble and $f'(x) = g'(x)$ $\forall x \in (a,b)$, Then $f(x) = g(x) + c$ for some constant $c \in \mathbb{R}$.

Pf: $(f-g)'(x) = 0$ $\forall x \in (a,b) \Rightarrow f - g = c$. $\square$

Cor 3: If $f : (a,b) \to \mathbb{R}$ is diff'ble and

$$\text{if } f'(x) \text{ is } \left\{ \begin{array}{l} \text{strictly pos.} \\ \text{positive} \\ \text{strictly neg} \\ \text{negative} \end{array} \right. \quad \forall x \in (a,b) \text{ then } f \text{ is } \left\{ \begin{array}{l} \text{strictly inc.} \\ \text{increasing} \\ \text{strictly dec.} \\ \text{decreasing} \end{array} \right.$$

Pf: For $a < x_1 < x_2 < b$, the sign of

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is determined by the sign of $f(x_2) - f(x_1)$, but the above $= f'(c)$ for some $c \in (x_1, x_2)$.

So e.g. if $f'(c) > 0 \Rightarrow f(x_2) - f(x_1) > 0$. $\square$

Remark: The converse is not true in general: $f(x) = x^3$ is strictly increasing but $f'(0) = 0$.

Taylor's Theorem V.4

Higher order derivatives: Suppose $f : U \to \mathbb{R}$ is diff'ble and that $f' : U \to \mathbb{R}$ is also diff'ble. We say f is twice differentiable and write

$$(f')' = f'' = \frac{d^2}{dx^2}(f) = \frac{d^2 f}{dx^2}$$

If $f'' : U \to \mathbb{R}$ is diff'ble, we say f is three times diff'ble and write

$$(f'')' = f^{(3)} = \frac{d^3}{dx^3}(f) = \frac{d^3 f}{dx^3}$$

In general, for $n \in \mathbb{N}$ we say f is n times

differentiable if $\overbrace{(\cdots((f')')\cdots)'}^{n} : U \to \mathbb{R}$ exists
and denote this by
$$f^{(n)} = \frac{d^n}{dx^n}(f) = \frac{d^n f}{dx^n}$$
we also call $f^{(n)}$ the nth derivative of f
in this way, it makes sense to write
$f^{(0)} = f$ and call f the zeroth derivative

11/17/2017

Recall factorial notation:
$$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$$

~~Lemma: Let $a, b \in \mathbb{R}$, $a<b$, and let $f:(a,b)\to\mathbb{R}$ be $(n+1)$-times diff'ble. For $x_1, x_2 \in (a,b)$ define $R_n(x_1,x_2)$ by~~
$$f(x_2) = f(x_1) + \frac{f'(x_1)}{1!}(x_2-x_1) + \frac{f''(x_1)}{2!}(x_2-x_1)^2 + \cdots + \frac{f^{(n)}(x_1)}{n!}(x_2-x_1)^n + R_n(x_1,x_2)$$
~~Then for any~~

Lemma Let $U \subseteq \mathbb{R}$ be an open interval and let $f:U\to\mathbb{R}$ be $(n+1)$-times diff'ble. For any $a,b \in U$ define $R_n(b,a) \in \mathbb{R}$ by:
$$f(b) = f(a) + \frac{f'(a)(b-a)}{1!} + \frac{f''(a)(b-a)^2}{2!} + \cdots + \frac{f^{(n)}(a)(b-a)^n}{n!} + R_n(b,a)$$

Then
$$\frac{d}{dx} R_n(b,x) = -\frac{f^{(n+1)}(x)(b-x)^n}{n!}$$

Pf: For $x \in U$, we have
$$R_n(b,x) = f(b) - f(x) - \frac{f'(x)}{1!}(b-x) - \cdots - \frac{f^{(n)}(x)}{n!}(b-x)^n$$

Since $R_n(b,x): U \to \mathbb{R}$ is a sum of diff'ble functions, it is diff'ble with: (using the product rule):

$$\frac{d}{dx} R_n(b,x) = 0 - f'(x) - \left[\frac{f''(x)(b-x)}{1!} - \frac{f'(x)(-1)}{1!}\right] - \cdots - \left[\frac{f^{(n+1)}(x)(b-x)^n}{n!} - \frac{f^{(n)}(x)\, n(-1)(b-x)^{n-1}}{n!}\right]$$

$$\text{(Telescoping)} = -\frac{f^{(n+1)}(x)(b-x)^n}{n!} \qquad \square$$

Thm (Taylor's Theorem)

Let $U \subseteq \mathbb{R}$ be an open interval, and let $f: U \to \mathbb{R}$ be $(n+1)$-times diff'ble. Then for any $a, b \in U$ $\exists\, c$ between a and b s.t.

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

Pf: ~~By the previous~~ Letting $R_n(b,a)$ be as in the previous lemma, we must show

$$R_n(b,a) = \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

for some c between a and b. Define

$$k := \frac{R_n(b,a)\,(n+1)!}{(b-a)^{n+1}} \in \mathbb{R}$$

so that

$$R_n(b,a) = k \cdot \frac{(b-a)^{n+1}}{(n+1)!}.$$

Consider $\varphi: U \to \mathbb{R}$ defined by

$$\varphi(x) = R_n(b,x) - k\frac{(b-x)^{n+1}}{(n+1)!}$$

Then $\varphi(a) = 0$ by def'n of k. Also, $\varphi(b) = 0$ by def'n of R_n. Moreover, by the previous Lemma, φ is diff'ble. Thus Rolle's thm $\Rightarrow \exists\, c$ between a + b s.t.

$$0 = \varphi'(c) = -\frac{f^{(n+1)}(c)(b-c)^n}{n!} + k\frac{(b-c)^n}{n!}$$

$$\iff k = f^{(n+1)}(c).$$

Hence $R_n(b,a) = f^{(n+1)}(c)\dfrac{(b-a)^{n+1}}{(n+1)!}$ $\square$

Remark: This theorem says that if f has enough derivatives, then we can approximate it in terms of polynomial and the error is determined by the next derivative. If we can control how big the derivatives get, we can say something stronger:

Lemma: For any $r \in \mathbb{R}$, $\lim_{n\to\infty} \frac{r^n}{n!} = 0$.

Pf: Let $k_1 \le |r| \le k_1+1$ $\quad k_2 - 1 \le 2|r| < k_2$ $\quad k_1, k_2 \in \mathbb{N}$.

Then for $k \in \{k_1+1, k_1+2, \ldots, k_2-1\}$ have: $\frac{|r|}{k} \le 1$

and for $k \ge k_2$, $\frac{|r|}{k} < \frac{1}{2}$.

Thus for $n \ge k_2$ have

$$\frac{|r|^n}{n!} = \frac{|r|}{n} \cdot \frac{|r|}{n-1} \cdots \frac{|r|}{k_2} \cdot \frac{|r|}{k_2-1} \cdots \frac{|r|}{k_1+1} \cdot \frac{|r|}{k_1} \cdots \frac{|r|}{1}$$

$$\le \left(\frac{1}{2}\right)^{n-k_2+1} \cdot 1^{k_2-k_1-1} \cdot M \xrightarrow{n\to\infty} 0 \quad \square$$

Prop Let $U \subseteq \mathbb{R}$ be an open interval, and let $f: U \to \mathbb{R}$ be n-times diff'ble $\forall n \in \mathbb{N}$. Further assume that $\exists R > 0$ s.t.

$$\max_{x \in U} |f^{(n)}(x)| \le R^n \quad \forall n \in \mathbb{N}.$$

Fix $a \in U$. Define $f_n: U \to \mathbb{R}$ by:

$$f_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \text{ polynomial}$$

Then $(f_n)_{n\in\mathbb{N}}$ conv. unif. to f on U.

Pf: Let $\varepsilon > 0$. Let $r = \text{length}(U)$. Then for any $x \in U$ we have $|x-a| < r$. Thus by Taylor's theorem:

$$|f(x)-f_n(x)| = \frac{|f^{(n+1)}(c)|}{(n+1)!}|x-a|^{n+1}$$

for some c between x and a. By assumption

$$\leq \frac{R^{n+1}\cdot r^{n+1}}{(n+1)!} = \frac{(R\cdot r)^{n+1}}{(n+1)!}$$

Since this doesn't depend on x, we have

$$\sup\{|f(x)-f_n(x)| : x\in U\} \leq \frac{(R\cdot r)^{n+1}}{(n+1)!} \xrightarrow{n\to\infty} 0$$

by Lemma. Thus $(f_n)_{n\in\omega}$ conv. unif. to f on U. $\square$

Riemann Integration VI

Def: Let $a,b\in\mathbb{R}$ with $a<b$. A partition of $[a,b]$ is a finite sequence $x_0, x_1, x_2, \dots, x_N$ s.t.

$$a = x_0 < x_1 < x_2 < \dots < x_N = b$$

The width of the partition $\{x_0, x_1, \dots, x_N\}$ is $\max\{x_i - x_{i-1} : i=1,\dots,N\}$

Def If $f:[a,b]\to\mathbb{R}$, a Riemann sum for f (corresponding to the partition ~~$\{x_0, x_1, \dots, x_N\}$~~ $a=x_0<x_1<\dots<x_N=b$) is the quantity:

$$f(x_1')\cdot(x_1-x_0)+f(x_2')\cdot(x_2-x_1)+\dots+f(x_N')(x_N-x_{N-1}) =: \sum_{i=1}^{N} f(x_i')(x_i-x_{i-1})$$

where $x_i'\in[x_{i-1},x_i]$ for each $i=1,\dots,N$. We call $x_1',\dots,x_N'$ representatives.

Note that each choice of $x_1',\dots,x_N'$ gives a different Riemann sums for f.