Note that $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$

$$\left| f(a + \tfrac{b-a}{n}) - y \right| < \varepsilon/2$$

Let $\delta > 0$ be s.t. if $x, y \in (a,b]$ satisfy

$$|x - y| < \delta,$$

then

$$|f(x) - f(y)| < \tfrac{\varepsilon}{2},$$

which exists by uniform cty of f on $(a,b]$.
Suppose $x \in (a,b]$ satisfies

$$|x - a| < \delta.$$

Let $n \in \mathbb{N}$ be s.t. $n \geq N$ and s.t.

$$a < a + \tfrac{b-a}{n} < x$$

Hence

$$\left| a + \tfrac{b-a}{n} - x \right| < \delta$$

and so

$$|f(x) - y| \leq \left| f(x) - f(a + \tfrac{b-a}{n}) \right| + \left| f(a + \tfrac{b-a}{n}) - y \right|$$
$$\leq \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

Thus $\lim_{x \to a} f(x) = y$. QED.

Continuous Functions on Connected Sets IV.5

Thm: Let (E, d) and (E', d') be metric spaces, and let $f: E \to E'$ be a cts function. If $S \subseteq E$ is connected, then so is $f(S)$.

Pf: We proceed by contrapositive. Suppose $f(S)$ is disconnected. Then $\exists$ disjoint, non-empty subsets $A, B \subseteq f(S)$ that are open rel. to $f(S)$ and $A \cup B = f(S)$. } dist

Define

$$A_1 := f^{-1}(A) \cap S \subseteq E$$
$$B_1 := f^{-1}(B) \cap S$$

We claim: (i) $A_1 \cap B_1 = \emptyset$

(ii) $A_1 \neq \emptyset \neq B_1$

(iii) $A_1 \cup B_1 = S$

(iv) A_1 and B_1 are open relative to S.

(i)–(iii) follow immediately from the corresponding properties of A and B. To see (iv), recall from the HW that A_1, B_1 open rel. to $f(S) \Rightarrow \exists$ open sets $U_A, U_B \subseteq E'$ s.t.

$$A = U_A \cap f(S)$$
$$B = U_B \cap f(S).$$

~~[illegible]~~ consequently

$$A_1 = f^{-1}(U_A) \cap S$$
$$B_1 = f^{-1}(U_B) \cap S$$

Since f is cts, $f^{-1}(U_A)$, $f^{-1}(U_B)$ are open in E. Appealing to the HW again, we obtain (iv). But (i)–(iv) $\Rightarrow$ S is disconnected $\square$.

~~Ex~~ ① Let $(\mathbb{R}^n, d_n)$ be n-dim'l Eucl. metric space. Let $L \subseteq \mathbb{R}^n$ be a line segment with endpoints

$$\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$$
$$\vec{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$$

Then L is connected. Indeed, $L = f([0,1])$ (or possibly $f((0,1))$, or $f((0,1])$, or $f([0,1))$ depending on if $\vec{x}, \vec{y} \in L$) where

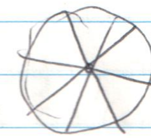
$$f(t) = ((1-t)x_1 + ty_1, \ldots, (1-t)x_n + ty_n).$$

f is cts since $\pi_1 \circ f, \ldots, \pi_n \circ f$ are cts and $[0,1]$ is connected.

② Let $(\mathbb{R}^n, d_n)$ be as before. Then for any $\vec{x} \in \mathbb{R}^n$ and $r > 0$

$$B(\vec{x}, r) \quad B[\vec{x}, r] \subseteq \mathbb{R}^n$$

are connected. Indeed, the ~~closed~~ balls is the unions of ~~diameters~~ radii: line segments passing through $\vec{x}$ and having endpoints on the boundary of the ball. Since each such line segment is connected by the previous example, and their common intersection is non-empty (it's $\{\vec{x}\}$) their union $B[\vec{x}, r]$ is connected by the HW.



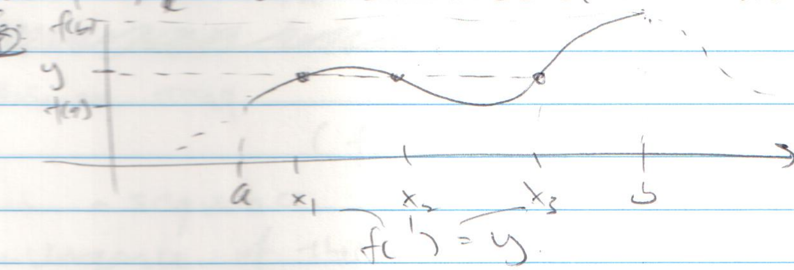
(great)^5-grandfa...

11/1/2017 State 3-equiv. def. of compactness, Heine-Borel, Bolzano-Weierstrass

Cor (Intermediate Value Theorem)

If $a, b \in \mathbb{R}$ satisfy $a < b$, and $f : [a,b] \to \mathbb{R}$ is cts, then for all y strictly in between $f(a)$ and $f(b)$, there is at least one $x \in (a,b)$ s.t. $f(x) = y$.

~~Pf~~:



Pf: Since $[a,b]$ is connected and f is cts, $f([a,b])$ is connected. ~~suppose~~ WLOG $f(a) < f(b)$.

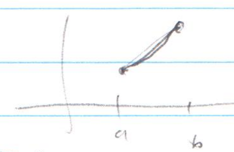
~~Suppose, towards a contradiction, $\exists\, y \in (f(a), f(b))$ s.t. $y \notin f[a,b]$. Then~~

~~$f([a,b]) = (f([a,b]) \cap (-\infty, y)) \cup (f([a,b]))$~~

We proved that if $\exists\, y \notin f([a,b])$ s.t. $f(a) < y < f(b)$, then $f([a,b])$ is disconnected. Thus $[f(a), f(b)] \subseteq f([a,b])$. $\square$

Ex: Let $a, b \in \mathbb{R}$ satisfy $a < b$. Suppose $f : [a,b] \to \mathbb{R}$ is cts, one-to-one, and $f(a) < f(b)$. Then

$$f([a,b]) = [f(a), f(b)]$$



The IVT $\Rightarrow$ $[f(a), f(b)] \subseteq f([a,b])$. Now, ~~suppose~~ for $y \in f([a,b])$, suppose towards a contradiction that $y \notin [f(a), f(b)]$. WLOG $f(b) < y$. We know

$y = f(x)$ for some $x \in (a,b)$. Applying the IVT to $f|_{[a,x]}$ we obtain $c \in (a,x)$ s.t. $f(c) = f(b)$. Since $c < x < b$, this contradicts f being 1-1. Thus $f([a,b]) \subseteq [f(a), f(b)]$ (QED).

Sequences of Functions IV.6

Let (E,d) and (E',d') be metric spaces. Given a collection of functions $\{f_n : n \in \mathbb{N}\}$ s.t.

$$f_n : E \to E' \quad \forall n \in \mathbb{N}$$

~~we can consider the sequence~~ for any $x \in E$, observe that

$$(f_n(x))_{n \in \mathbb{N}} \subseteq E'$$

is a sequence. We might wonder about convergence of this sequence and whether it holds for all $x \in E$ or only some $x \in E$. We will think of the collection of functions as a sequence itself:

$$(f_n)_{n \in \mathbb{N}}.$$

Def: Let (E,d) and (E',d') be metric spaces, and for each $n \in \mathbb{N}$ let $f_n : E \to E'$ be a function. For $x \in E$, we say $(f_n)_{n \in \mathbb{N}}$ converges at x if the sequence $(f_n(x))_{n \in \mathbb{N}} \subseteq E'$ converges. ~~We say (f_n) converges pointwise~~ For $S \subseteq E$, we say $(f_n)_{n \in \mathbb{N}}$ converges pointwise on S if it converges at every $x \in S$. If $S = E$, we simply say $(f_n)_{n \in \mathbb{N}}$ converges pointwise, or is pointwise convergent. In this case the function $f : E \to E'$ defined by

$$f(x) = \lim_{n \to \infty} f_n(x)$$