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Limit Arithmetic IV.3

- We explore how continuity & limits behave for functions valued in our favorite metric space: $\mathbb{R}$.

- Given a metric space (E,d) and considering $\mathbb{R}$ w/ the usual metric, we consider real-valued functions

$$f,g : E \to \mathbb{R}.$$

We can then consider:

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(f \cdot g)(x) = f(x) \cdot g(x)$$
$$(f/g)(x) = f(x)/g(x) \quad (\text{provided } g(x) \neq 0).$$

Prop Let (E,d) be a metric space and equip $\mathbb{R}$ w/ the usual metric. Suppose

$$f,g : E \to \mathbb{R}$$

are cts at $x_0 \in E$. Then so are $f+g$, $f-g$, & $f \cdot g$. If $g(x_0) \neq 0$ and $\exists r > 0$ s.t. $g(x) \neq 0$ for all $x \in B(x_0, r)$, then f/g is also cts at x_0.

Pf: Use charact. of cty in terms of sequences and apply previous Prop for sequences. $\square$

Cor: Let $x_0 \in E$ be a cluster point of E. If $f,g : E \to \mathbb{R}$ both have limits at x_0, then

$$\lim_{x \to x_0} (f+g)(x) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x)$$
$$\lim_{x \to x_0} (f-g)(x) = \lim_{x \to x_0} f(x) - \lim_{x \to x_0} g(x)$$
$$\lim_{x \to x_0} (f \cdot g)(x) = \lim_{x \to x_0} f(x) \cdot \lim_{x \to x_0} g(x)$$

If $\lim_{x\to x_0} g(x) \neq 0$, then $\lim_{x\to x_0} (f/g)(x) = \dfrac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)}$

For $n\in\mathbb{N}$ we now consider special functions

$$\pi_j : \mathbb{R}^n \to \mathbb{R} \qquad j=1,\dots,n$$

~~(where $\mathbb{R}^n$ and $\mathbb{R}$ are equipped Euclidean metrics)~~

defined by

$$\pi_j(x_1, x_2, \dots, x_n) := x_j$$

π_j is called the jth-coordinate ~~component~~ function/projection.

Lemma: ~~If~~ with $(\mathbb{R}^n, d)$ the n-dim'l Euc metric space and $\mathbb{R}$ equipped with the usual metric, $\pi_j : \mathbb{R}^n \to \mathbb{R}$ is cts.

Pf: We have essentially observed this previously many times. Fix

$$\vec{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$$

and let $\varepsilon > 0$. Setting $\delta = \varepsilon$, if $\vec{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$ satisfies

$$d(\vec{x}, \vec{y}) < \delta$$

then

$$|\pi_j(\vec{x}) - \pi_j(\vec{y})| = |x_j - y_j| = \sqrt{(x_j - y_j)^2}$$
$$\leq \sqrt{(x_1-y_1)^2 + \dots + (x_n - y_n)^2}$$
$$= d(\vec{x}, \vec{y}) < \delta = \varepsilon.$$

Hence π_j is cts at $\vec{x}$. $\square$

Prop: Let (E, d) be a metric space and let $(\mathbb{R}^n, d_n)$ be n-dim'l Euclidean metric space. Then

$$f : E \to \mathbb{R}^n$$

is cts at $x_0 \in E$ if and only if

$$\pi_j \circ f : E \to \mathbb{R}$$

is cts at x_0 for each $j = 1, \ldots, n$.

Pf ($\Longrightarrow$) If f is cts at x_0, then $\pi_j \circ f$ is cts at x_0 for each $j = 1, \ldots, n$ as the composition of two cts functions.

($\Longleftarrow$) Suppose each $\pi_1 \circ f, \ldots, \pi_n \circ f$ are cts at x_0. Let $\varepsilon > 0$. Then for each $j = 1, \ldots, n$, $\exists\, \delta_j > 0$ s.t.

$$* \qquad |\pi_j \circ f(x) - \pi_j \circ f(x_0)| < \frac{\varepsilon}{\sqrt{n}}$$

whenever $x \in E$ satisfies $d(x, x_0) < \delta_j$.

Set $\delta = \min\{\delta_1, \ldots, \delta_n\}$. If $x \in E$ satisfies $d(x, x_0) < \delta$, then ($*$) holds for each coordinate $j = 1, \ldots, n$. Hence

$$d_n(f(x), f(x_0)) = \sqrt{(\pi_1 \circ f(x) - \pi_1 \circ f(x_0))^2 + \cdots + (\pi_n \circ f(x) - \pi_n \circ f(x_0))^2}$$

$$< \sqrt{\frac{\varepsilon^2}{n} + \cdots + \frac{\varepsilon^2}{n}} = \varepsilon.$$

Thus f is cts at x_0. $\square$

Continuous Functions on Compact Sets, IV.4

Thm: Let (E, d) and (E', d') be metric spaces, and let $f: E \to E'$ be a cts function. If $S \subseteq E$ is compact, then so is $f(S)$.

Pf: Let $\{U_i\}_{i \in I}$ be an open cover of $f(S) \subseteq E'$. Since f is cts,

$$f^{-1}(U_i) \subseteq E$$

is open for each $i \in I$. Moreover,

$$S \subseteq \bigcup_{i \in I} f^{-1}(U_i)$$

Indeed, if $x \in S$, then $f(x) \in f(S)$ and so $\exists\, i_0 \in I$ s.t. $f(x) \in U_{i_0}$. But then $x \in f^{-1}(U_{i_0})$. So the claimed inclusion holds.

We have shown that $\{f^{-1}(U_i)\}_{i \in I}$