so combining $(*)$ and $(**)$ we have

$$|d(x,y)-d(y,z)| \le d(x,z) \qquad \square$$

Open and closed sets (III.2)

Def: For (E,d) a metric space, $x \in E$, and $r>0$ we define the open ball in E of center x and radius r as the set

$$B(x,r) := \{y \in E : d(x,y) < r\}$$

9/11/2017 We define the closed ball in E of center x and radius r as the set

$$B[x,r] := \{y \in E : d(x,y) \le r\}$$

We'll say "ball" to refer to an either open or closed ball, when the distinction does not matter.

Picture: (27)

- Note that since $d(x,x)=0$, we always have

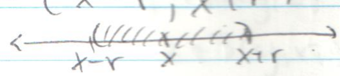
$$x \in B(x,r) \qquad x \in B[x,r]$$

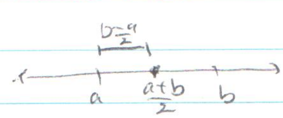
for any $r>0$.

Ex: • $(E,d) = (\mathbb{R}, |\cdot|)$

$$B(x,r) = \{y \in \mathbb{R} : |x-y| < r\} \quad x-r < y < x+r$$

$$= (x-r, x+r) \quad \text{interval notation}$$





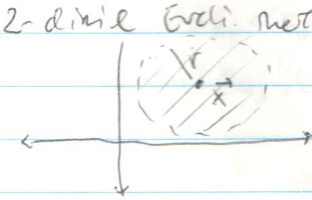
Also, the interval (a,b):

So $(a,b) = B\left(\frac{a+b}{2}, \frac{b-a}{2}\right)$. $B[x,r] = [x-r, x+r]$

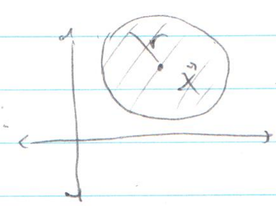
$[a,b] = B\left[\frac{a+b}{2}, \frac{b-a}{2}\right]$

- $(E,d) = (\mathbb{R}^2$, 2-dim'l Eucli. metric$)$

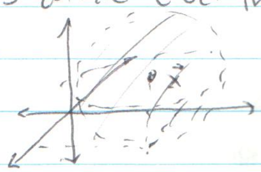
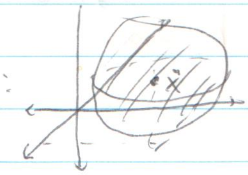
$B(\vec{x},r)$



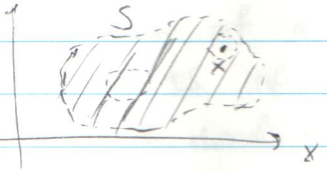
$B[x,r]$:

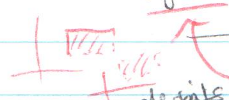


$\cdot (E,d) = (\mathbb{R}^3, \text{3-dim'l Euc. metric})$

$B(\vec{x},r)$:  $B[\vec{x},r]$: 

Def: A subset $S \subset E$ is open if for all $s \in S$ $\exists\, r>0$ s.t. $B(s,r) \subseteq S$ (r may depend on S)

Picture: open:  not open 

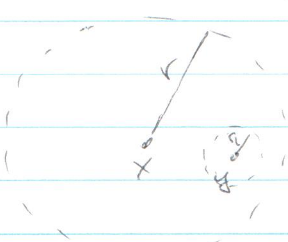


details

Prop: In a metric space (E,d), any open ball is open.

Pf: Let $x \in E$, $r>0$. We'll show $S=B(x,r)$ is open. Let $y \in B(x,r)$. We must find some $r_1>0$ s.t.

$$B(y,r_1) \subseteq B(x,r)$$



Let $r_1 = r - d(x,y)$. Note $d(x,y)<r$ since $y \in B(x,r)$. So $r_1>0$. Suppose $z \in B(y,r_1)$. We'll need to show $z \in B(x,r)$.

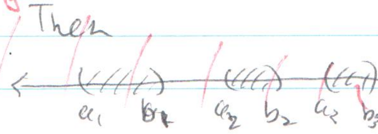
That is, we must show $d(x,z)<r$. Using the triangle inequality:

$$d(x,z) \le d(x,y) + d(y,z) < d(x,y) + r_1 = r$$

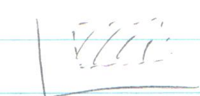
So $z \in B(x,r)$. Since $z \in B(y,r_1)$ was arbitrary we have $B(y,r_1) \subseteq B(x,r)$. Since $y \in B(x,r)$ was arbitrary, we see that $B(x,r)$ is open. □

$S=(a,b)$ open

Ex (1) In $(\mathbb{R}, |\cdot|)$ Let $a_1 < b_1 < a_2 < b_2 < a_3 < b_3$. Then

$S = (a_1,b_1) \cup (a_2,b_2) \cup (a_3,b_3)$ 

is open, $[a,b]$ not open

(2) In $(\mathbb{R}^2, \text{2-dim'l Euc metric})$ $S = (1,3) \times (1,2)$ 

is open. $\boxed{}$ – non ex.

(3) In $\mathbb{R}^2$ w/ same metric $S = \mathbb{R}^2 \setminus \{(0,0)\}$ is open

(4) for $\{(x,y)\} \in \mathbb{R}^2$, $S = \{(x,y)\}$ is not open.

9/13/2017 Proposition Let (E,d) be a metric space. Then

(i) the subset $\emptyset$ is open;

(ii) the "subset" E is open;

(iii) the union of any collection of open subsets of E is open.

(iv) the intersection of a finite collection of open subsets of E is open.

Pf: (i) The condition we need to check holds vacously.

(ii) For any $x \in E$ and any $r>0$, $B(x,r) \subset E$ by def of $B(x,r)$. So E is open

(iii) Let $\{U_i\}_{i \in I}$ be a collection of open subsets $U_i \subseteq E$. Let $U = \bigcup_{i \in I} U_i$. Let $x \in U$. Then $\exists i \in I$ s.t. $x \in U_i$. Since U_i is open, $\exists r > 0$ s.t. $B(x,r) \subseteq U_i$. But then $B(x,r) \subseteq \bigcup_{i \in I} U_i = U$. Hence U is open.

(iv) Let $U_1, \dots, U_n$ be open subsets of E. for some $n \in \mathbb{N}$. Let

$$V = U_1 \cap U_2 \cap \cdots \cap U_n$$

and let $x \in V$. Then $x \in U_i$ for each $i \in 1, \dots, n$. Since each U_i is open, $\exists r_i > 0$ for each $i = 1, \dots, n$ s.t.

$$B(x, r_i) \subseteq U_i.$$

~~Let~~ Take $r = \min\{r_1, \dots, r_n\} > 0$. Then

$$B(x,r) \subseteq B(x, r_i) \subseteq U_i \quad \forall i = 1, \dots, n.$$

Thus $B(x,r) \subseteq U_1 \cap \cdots \cap U_n = V$. So we have shown V is open $\square$

Remark The word "finite" in (iv) is necessary. An infinite intersection of open sets need not be open.

Ex: Let $U_n = B(0,\frac{1}{n})$ in $(\mathbb{R}, |\cdot|)$. Then the infinite intersection

$$\bigcap_{n\in\mathbb{N}} U_n = \{0\}$$

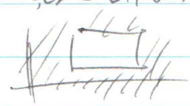
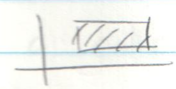
is not open. Indeed $\forall r>0$, $B(0,r) \not\subseteq \{0\}$ since $\frac{r}{2} \in B(0,r)$ for example.

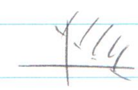
Def: A subset $S \subseteq E$ is closed if its complement

$$S^c = \{x\in E : x \notin S\} = E \setminus S$$

is open.

Ex: (1) $(-\infty,0] \cup [1,\infty)$  is closed.

(2)  

(3)  $y \geq mx+b$ closed,  $y > mx+b$ open

(4) $\{\vec{x}\} \subseteq \mathbb{R}^n$.

Prop: Let (E,d) be a metric space. For any $x\in E$ and any $r\geq 0$, the closed ball $B[x,r]$ is closed.

Pf: Let $x\in E$ and $r>0$. Since

$$B[x,r] = \{y\in E : d(x,y) \leq r\}$$

we have

$$B[x,r]^c = \{y \in E : d(x,y) > r\}.$$

We need to show it is open. Let $y \in B[x,r]^c$.
We need to find $r_1 > 0$ s.t. $B(y,r_1) \subseteq B[x,r]^c$.
Since $d(x,y) > r$, we can let $r_1 = d(x,y) - r > 0$.
Then for any $z \in B(y,r_1)$, the reverse triangle inequality implies

$$\begin{aligned} d(x,z) &\geq |d(x,y) - d(y,z)| \\ &\geq d(x,y) - d(y,z) \\ &> d(x,y) - r_1 = r \end{aligned}$$

so $d(x,z) > r \Rightarrow z \in B[x,r]^c \Rightarrow B(y,r_1) \subseteq B[x,r]^c$.
So $B[x,r]^c$ is open, which means $B[x,r]$ is closed. $\square$

<u>Proposition</u> Let (E, d) be a metric space. Then
(i) the subset ϕ is closed;
(ii) the subset E is closed;
(iii) the intersection of any collection of closed subsets of E is closed;
(iv) the union of a <u>finite</u> number of closed subsets of E is closed.

9/15/2017 <u>Pf</u>: (i) Since $\phi^c = E$, and E is open, we get that ϕ is closed.
(ii) Since $E^c = \phi$, and ϕ is open, we get that E is closed.
(iii) Let $\{V_i\}_{i \in I}$ be a collection of closed subsets of E. Then

$$\left(\bigcap_{i \in I} V_i\right)^c = \left(\bigcup_{i \in I} V_i^c\right)$$

Note that each V_i^c is open, so by previous prop, $\bigcup_{i \in I} V_i^c$ is open. Hence $\bigcap_{i \in I} V_i$ is closed.
(iv) For closed subsets $V_1, \ldots, V_n \subseteq E$,

$$(V_1 \cup \cdots \cup V_n)^c = V_1^c \cap \cdots \cap V_n^c$$

and the latter is a finite intersection of open subsets. So it is open by the previous prop, which implies $V_1 \cup \cdots \cup V_n$ is closed. $\square$

<u>Remark</u>: The word "finite" in (iv) is once more important.

<u>Ex</u>: For each $n \in \mathbb{N}$, let $V_n = [\frac{1}{n}, 1 - \frac{1}{n}]$. Then the infinite union

$$\bigcup_{n \in \mathbb{N}} V_n = (0, 1)$$

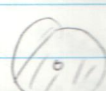
is open.

Warning: Sets can be open, closed, ~~open~~ both, or neither.

Ex: • $\emptyset$ and E are both open and closed (clopen)
• In $(\mathbb{R}, |\cdot|)$, the set $[0,1)$ is neither.
• In $(\mathbb{R}^2$, 2-dim'l Euc metric)

$= \{\vec{y} \in \mathbb{R}^2 : r_1 < d(\vec{x},\vec{y}) \le r_2\}$ is neither

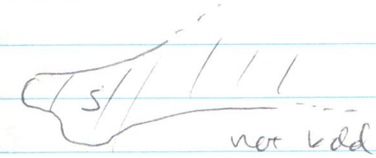
In particular



$= \{\vec{y} \in \mathbb{R} : 0 < d(\vec{x},\vec{y}) \le r_2\} = B[\vec{x}, r_2] \setminus \{\vec{x}\}$ is neither.

Ex: (1) $\mathbb{Z}, \mathbb{N}, \subseteq \mathbb{R}$ open or closed?
(2) $\mathbb{Q}$?

Def: A subset S of a metric space (E,d) is bounded if it is contained in some ball. That is, if $\exists\, x \in E$ and $r > 0$ s.t. $S \subseteq B(x,r)$



Ex: • In $(\mathbb{R}, |\cdot|)$ the interval $[0,1)$ is bdd. Contained in ball $B(0,100)$
• The interval $[0,+\infty)$ is not bdd.

• Observe that for $E = \mathbb{R}$, if S is bdd and $S \subseteq B(x,r)$ for some $x \in \mathbb{R}$, $r > 0$. Then $\forall s \in S$, $|x-s| < r \iff x-r < s < x+r$. That is, $x+r$ is an upper bound for S and $x-r$ is a lower bound.

Proposition: Let S be a non-empty, closed subset of $\mathbb{R}$. If S is bounded from above, then $\sup(S) \in S$. If S is bounded from below, then $\inf(S) \in S$

Pf: Suppose is bounded from above. Then $\sup(S)$ exists. Suppose, towards a contradiction, that $\sup(S) \notin S$. Then $\sup(S) \in S^c$, and we note that S^c is open since S is closed. Thus $\exists r > 0$ s.t. $B(\sup(S), r) \subseteq S^c$. However, one of the first things we proved about supremums was that we can always find $s \in S$ s.t.

$$\sup(S) - r < s \leq \sup(S).$$

This means $|s - \sup(S)| < r \Rightarrow s \in B(\sup(S), r)$, which contradicts $B(\sup(S), r) \subseteq S^c$. Thus, we must have $\sup(S) \in S$.

Next, if S is bounded from below, then $-S$ is bounded from above. Applying the preceeding argument to $-S$ we have

$$\sup(-S) \in -S.$$

From Homework, we have $\sup(-S) = -\inf(S)$ and so $-\inf(S) \in -S \Rightarrow \inf(S) \in S$. $\square$

Convergent Sequences III.3

Def: In a metric space (E, d) a sequence is an infinite ordered list of points in E: $(x_n)_{n \in \mathbb{N}} = (x_1, x_2, x_3, \ldots)$ for $x_n \in E$.

We want to formalize the notion of a limit. That is, given a sequence $(x_n)_{n \in \mathbb{N}} \subseteq E$