LIMIT SUPERIOR AND LIMIT INFERIOR

Throughout this note, $(x_n)_{n\in\mathbb{N}}$ will be a bounded sequence in $\mathbb{R}$.

Recall that we defined the limit superior and limit inferior in class as follows:

$$\limsup_{n\to\infty} x_n := \lim_{k\to\infty} \sup\{x_n \colon n \geq k\}$$
$$\liminf_{n\to\infty} x_n := \lim_{k\to\infty} \inf\{x_n \colon n \geq k\}.$$

However, Rosenlicht provides different definitions in Exercise 18 of Chapter III. To differentiate between the version we defined in class and the book version, we will adopt the following notation for the book's versions:

$$\overline{\lim_{n\to\infty}}\, x_n := \sup\{y \in \mathbb{R} \colon x_n > y \text{ for infinitely many } n \in \mathbb{N}\}$$
$$\underline{\lim_{n\to\infty}}\, x_n := \inf\{y \in \mathbb{R} \colon x_n < y \text{ for infinitely many } n \in \mathbb{N}\}.$$

Our hope is that

$$\limsup_{n\to\infty} x_n = \overline{\lim_{n\to\infty}}\, x_n$$
$$\liminf_{n\to\infty} x_n = \underline{\lim_{n\to\infty}}\, x_n.$$

We shall prove the first equality, with the second equality holding by a similar argument.

First we establish some notation. For $k \in \mathbb{N}$, denote $A_k := \sup\{x_n \colon n \geq k\}$. Also, let

$$S := \{y \in \mathbb{R} \colon x_n > y \text{ for infinitely many } n \in \mathbb{N}\}.$$

Given $y \in S$ and any $k \geq \mathbb{N}$, there exists $n \geq k$ such that $x_n > y$. Indeed, since there are infinitely many $x_n > y$, one of them must have an index satisfying $n \geq k$. And when $n \geq k$ we have

$$y < x_n \leq A_k.$$

This means that for each $k \in \mathbb{N}$, A_k is an upper bound for S. Consequently,

$$\overline{\lim_{n\to\infty}}\, x_n = \sup(S) \leq A_k \qquad \forall k \in \mathbb{N},$$

and so this inequality also holds when we take the limit as $k \to \infty$ on the right-hand side:

$$\overline{\lim_{n\to\infty}}\, x_n \leq \lim_{k\to\infty} A_k = \limsup_{n\to\infty} x_n.$$

To see the other inequality, we will show that for all $\epsilon > 0$

$$\limsup_{n\to\infty} x_n - \epsilon \in S.$$

To do so, for each fixed $\epsilon > 0$ we must find an infinite number of sequence elements x_n which are strictly greater than the above quantity. Fix $\epsilon > 0$, then for any $k \in \mathbb{N}$, since A_k is the supremum of the set $\{x_n \colon n \geq k\}$, we know there exists $n \geq k$ such that

$$A_k - \epsilon < x_n \leq A_k.$$

Since A_k is a decreasing sequence, this in particular implies

$$x_n > A_k - \epsilon \geq \lim_{k\to\infty} A_k - \epsilon = \limsup_{n\to\infty} x_n - \epsilon.$$

When $k = 1$, denote the subscript n we obtain in this way by n_1. When $k = n_1 + 1$, denote the subscript n we obtain in this way by n_2. Note that $n_2 \geq k = n_1 + 1 > n_1$. Iterating this process (i.e. by induction) we obtain a strictly increasing sequence of subscripts $n_1 < n_2 < \cdots$ such that

$$x_{n_\ell} > \limsup_{n\to\infty} x_n - \epsilon \qquad \forall \ell \in \mathbb{N}.$$

Hence there are infinitely many sequence elements strictly greater than the number $\limsup_{n\to\infty} x_n - \epsilon$, which implies $\limsup_{n\to\infty} x_n - \epsilon \in S$. Consequently,

$$\limsup_{n\to\infty} x_n - \epsilon \leq \sup(S) = \overline{\lim_{n\to\infty}} x_n,$$

or

$$\limsup_{n\to\infty} x_n \leq \overline{\lim_{n\to\infty}} x_n + \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we in fact have

$$\limsup_{n\to\infty} x_n \leq \overline{\lim_{n\to\infty}} x_n.$$

In conjunction with the previous inequality, we have the desired equality.