The Lebesgue Integral

Brent Nelson

In these notes we give an introduction to the Lebesgue integral, assuming only a knowledge of metric spaces and the Riemann integral. For more details see [1, Chapters 1 and 2]

1 Measures

Before we can discuss the the Lebesgue integral, we must first discuss "measures." Given a set X, a measure is, loosely-speaking, a map that assigns sizes to subsets of X. We will want such a map to satisfy certain intuitive properties (such as the size of a disjoint union is the sum of the individual sizes), and consequently only some subsets will be in the domain of the measure. These desired properties (see Definition 1.6) motivate the definition of "σ-algebras."

1.1 σ-algebras

Definition 1.1. Given a non-empty set X, a **σ-algebra on** X is a non-empty collection $\mathcal{M}$ of subsets of X such that

(i) $S \in \mathcal{M}$ implies $S^c \in \mathcal{M}$ (closed under complements),

(ii) $S_i \in \mathcal{M}$ for $i \in I$ with I a countable index set implies $\bigcup_{i\in I} S_i \in \mathcal{M}$ (closed under countable unions).

Observe that for any $S \in \mathcal{M}$, we have $X = S \cup S^c$ so that $X \in \mathcal{M}$. Then also $\emptyset \in \mathcal{M}$ since $\emptyset = X^c$. Furthermore, $\mathcal{M}$ is closed under countable intersections since

$$\bigcap_{i\in I} S_i = \left(\bigcup_{i\in I} S_i^c\right)^c.$$

Example 1.2. For an arbitrary non-empty set X, one can always define the **trivial σ-algebra** $\mathcal{M} = \{\emptyset, X\}$. This is usually too small to be an interesting σ-algebra.

Example 1.3. For an arbitrary non-empty set X, the **power set of** X is the collection $\mathcal{P}(X)$ of **all** subsets of X. This is easily seen to be a σ-algebra, but it is usually too large to be a useful σ-algebra (at least with respect to measures).

Definition 1.4. For a non-empty set X and a collection $\mathcal{F}$ of subsets of X, the **σ-algebra generated by** $\mathcal{F}$ is the smallest σ-algebra $\mathcal{M}$ such that $\mathcal{F} \subset \mathcal{M}$, and is denoted by $\mathcal{M}(\mathcal{F})$.

An equivalent definition for $\mathcal{M}(\mathcal{F})$ is the intersection of all σ-algebras containing $\mathcal{F}$. It is an easy exercise to verify that the intersection of two σ-algebras is again a σ-algebra. Since there always exists at least one σ-algebra containing $\mathcal{F}$ (namely the power set of X: $\mathcal{F} \subset \mathcal{P}(X)$), $\mathcal{M}(\mathcal{F})$ always exists.

Example 1.5. Suppose $X = E$ a metric space with metric d. Then we can consider the collection $\mathcal{F}$ of open subsets of X (for X a general topological space we can consider the same collection). The σ-algebra $\mathcal{M}(\mathcal{F})$ generated by this collection is known as the **Borel σ-algebra** and is denoted $\mathcal{B}_X$. Note that since $\mathcal{B}_X$ is closed under complements, it also contains all closed subsets of X.

Actually, $\mathcal{B}_X$ is generally much larger than this: it contains all countable intersections of open sets (which are called G_δ sets), all countable unions of closed sets (called F_σ sets), countable unions of G_δ sets (called $G_{\delta\sigma}$ sets and so on.

1.2 Measures and measure spaces

Definition 1.6. Given a non-empty set X equipped with a σ-algebra $\mathcal{M}$ on X, a **measure on** $\mathcal{M}$ is a map $\mu\colon \mathcal{M} \to [0, \infty]$ such that

(i) $\mu(\emptyset) = 0$,

(ii) for $\{S_i\}_{i\in I}$ a countable collection of disjoint subsets $S_i \in \mathcal{M}$, $\mu\left(\bigcup_{i\in I} S_i\right) = \sum_{i\in I} \mu(S_i)$.

The triple $(X, \mathcal{M}, \mu)$ is called a **measure space** and the elements of $\mathcal{M}$ are called **measurable sets**. In particular, $S \subset \mathbb{R}$ is said to be **Borel measurable** if $S \in \mathcal{B}_{\mathbb{R}}$ (see Example 1.5).

Note that since the summation in (ii) is of positive terms, the order of summation does not matter.

Example 1.7. Given a non-empty set X equipped with a σ-algebra $\mathcal{M}$, we can always define the zero measure by $\mu(S) = 0$ for all $S \in \mathcal{M}$.

Example 1.8. Let $X = \mathbb{R}$ which we think of as a metric space equipped with the metric $d(x, y) = |x - y|$. Then let $\mathcal{B}_{\mathbb{R}}$ be the Borel σ-algebra in Example 1.5. Let $F\colon \mathbb{R} \to \mathbb{R}$ be an increasing, continuous function. Then one can define a measure μ_F on $\mathcal{B}_{\mathbb{R}}$ by

$$\mu_F(S) := \inf\left\{ \sum_{j=1}^{\infty} [F(b_j) - F(a_j)] \colon \bigcup_{j=1}^{\infty} (a_j, b_j] \supset S \right\}.$$

That is, given a set $S \in \mathcal{B}_{\mathbb{R}}$, consider all the ways to cover S by countable unions of disjoint half-open intervals $\bigcup_{j=1}^{\infty}(a_j, b_j]$. This larger set is given the measure $\mu_F\left(\bigcup_{j=1}^{\infty}(a_j, b_j]\right) = \sum_{j=1}^{\infty}[F(b_j) - F(a_j)]$. Taking the infimum of this quantity over all such coverings gives the measure for S. In particular, for any interval $(a, b]$ one has $\mu_F((a, b]) = F(b) - F(a)$. This equality also holds if $(a, b]$ is replaced with (a, b), $[a, b]$, or $[a, b)$ though this requires some checking. That this defines a measure is also not immediate, but not beyond your capability to check.

The following are some useful general properties of measures, the proofs of which are straightforward.

Theorem 1.9. *Let $(X, \mathcal{M}, \mu)$ be a measure space.*

*(a) (**Monotonicity**) If $S, T \in \mathcal{M}$ with $S \subset T$, then $\mu(S) \leq \mu(T)$.*

*(b) (**Subadditivity**) If $\{S_i\}_{i\in I} \subset \mathcal{M}$ is a countable collection of not necessarily disjoint subsets of X, then*

$$\mu\left(\bigcup_{i\in I} S_i\right) \leq \sum_{i\in I} \mu(S_i).$$

*(c) (**Continuity from below**) If $\{S_n\}_{n\in\mathbb{N}} \subset \mathcal{M}$ is an ascending chain $S_1 \subset S_2 \subset S_3 \subset \cdots$, then*

$$\mu\left(\bigcup_{n=1}^{\infty} S_n\right) = \lim_{n\to\infty} \mu(S_n).$$

*(d) (**Continuity from above**) If $\{S_n\}_{n\in\mathbb{N}} \subset \mathcal{M}$ is a descending chain $S_1 \supset S_2 \supset S_3 \supset \cdots$ and $\mu(S_1) < \infty$, then*

$$\mu\left(\bigcap_{n=1}^{\infty} S_n\right) = \lim_{n\to\infty} \mu(S_n).$$

1.3 Null sets and complete measures

Definition 1.10. Given a measure space $(X, \mathcal{M}, \mu)$, a subset $N \in \mathcal{M}$ such that $\mu(N) = 0$ is called a **μ-null set** (or just a **null set** when the measure is clear from the context). If some property is true for all points $x \in X$ except on a null set, we say the property holds **μ-almost everywhere** (or just **almost everywhere**).

Because of the monotonicity condition, if $N \in \mathcal{M}$ is a null set and $S \subset N$ is also measurable ($S \in \mathcal{M}$), then $\mu(S) = 0$. However, it need not be the case that any subset of a null set is measurable. When this *is* the case, we have the following definition:

Definition 1.11. A measure space $(X, \mathcal{M}, \mu)$ is said to be **complete** if for any μ-null set N, all subsets $S \subset N$ are also measurable (and consequently also μ-null sets).

Just as with metric spaces, a measure space $(X, \mathcal{M}, \mu)$ that is not *a priori* complete has a completion $(X, \overline{\mathcal{M}}, \overline{\mu})$. That is, $(X, \overline{M}, \overline{\mu})$ is a complete measure space such that $\mathcal{M} \subset \overline{\mathcal{M}}$ and $\overline{\mu}(S) = \mu(S)$ for all $S \in \mathcal{M}$. It is via this completion that we obtain the Lebesgue measure.

1.4 The Lebesgue measure

Definition 1.12. Define $F \colon \mathbb{R} \to \mathbb{R}$ by $F(x) = x$, and let μ_F be as in Example 1.8. Let $(\mathbb{R}, \mathcal{L}, m)$ be the completion of the measure space $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu_F)$. Then m is called the **Lebesgue measure**, and subsets $S \subset \mathbb{R}$ are said to be **Lebesgue measurable** if $S \in \mathcal{L}$.

Recall that for any half-open interval $(a, b]$, $\mu_F((a, b]) = F(b) - F(a) = b - a$ when $F(x) = x$. Since the Lebesgue measure extends this measure, we also have $m((a, b]) = b - a$ (and similarly for $[a, b)$, $[a, b]$, and (a, b)). Thus the Lebesgue measure is, for sufficiently nice subsets of $\mathbb{R}$ like the intervals, simply giving what we usually think of as the length. In particular, $m(\mathbb{R}) = \infty$ and $m(\{x\}) = 0$ for any $x \in \mathbb{R}$. However, some sets which we might think of as "large" are actually quite small according to m.

Example 1.13. Let $S \subset \mathbb{R}$ be any countable subset. In this case we can enumerate $S = \{x_n\}_{n \in \mathbb{N}}$. Since the singleton sets $\{x_n\}$ are closed for each $n \in \mathbb{N}$, we have $\{x_n\} \subset \mathcal{B}_{\mathbb{R}} \subset \mathcal{L}$. As a σ-algebra, $\mathcal{B}_{\mathbb{R}}$ is closed under countable unions and hence

$$S = \bigcup_{n \in \mathbb{N}} \{x_n\} \in \mathcal{B}_{\mathbb{R}} \subset \mathcal{L}.$$

Thus S is Borel measurable and consequently Lebesgue measurable. So, m is defined on the input S, and we claim that $m(S) = 0$. Indeed, let $\epsilon > 0$ and for each $n \in \mathbb{N}$ define $a_n := x_n - \frac{\epsilon}{2^{n+1}}$ and $b_n := x_n + \frac{\epsilon}{2^{n+1}}$. Then $S \subset \bigcup_{n=1}^{\infty} (a_n, b_n]$ and thus

$$m(S) \leq \sum_{n=1}^{\infty} m((a_n, b_n]) = \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon.$$

Since $m(S) \geq 0$ and $\epsilon > 0$ was arbitrary, we have that $m(S) = 0$.

In particular, $m(\mathbb{Q}) = 0$; that is, the rationals are negligible with respect to the Lebesgue measure, despite being an infinite set. We had previously seen that $\mathbb{R}$ is uncountable (and consequently so is the set of irrationals $\mathbb{R} \setminus \mathbb{Q}$), which gave some indication of how much larger $\mathbb{R}$ is than $\mathbb{Q}$. Here we are seeing this comparison taken to the extreme: $m(\mathbb{R}) = \infty$ while $m(\mathbb{Q}) = 0$. Note also that by Definition 1.6.(ii), $\mu(\mathbb{R} \setminus \mathbb{Q}) = \mu(\mathbb{R}) - \mu(\mathbb{Q}) = \infty$.

Another consequence of this fact will be that the "Lebesgue integral" of a function f will be invariant under countable modifications to the definition of f. Recall that for the Riemann integral of a function f, we could modify the value of f at a *finite* number of points and still obtain the same integral. With the more robust Lebesgue integral, we will have the same property but with "finite" upgraded to "countable".

Given that $\mathcal{L}$ contains all open and closed subsets of $\mathbb{R}$ (along with all G_δ subsets, F_σ subsets, $G_{\delta\sigma}$ subsets, etc.), it is perfectly reasonable to wonder if there are any subsets $S \subset \mathbb{R}$ such that $S \notin \mathcal{L}$. That is, do non-Lebesgue measurable subsets exist?

Proposition 1.14. *Define an equivalence relation on $[0, 1)$ such that $x \sim y$ if and only if $x - y \in \mathbb{Q}$. Let $N \subset [0, 1)$ contain exactly one representative from each equivalence class. Then $N \notin \mathcal{L}$.*

Proof. Suppose, towards a contradiction, that $N \in \mathcal{L}$ so that m is defined on N. It is easily checked that m is invariant under translations. That is, if $S \in \mathcal{L}$ and $x \in \mathbb{R}$, then $S + x \in \mathcal{L}$ with $m(S + x) = m(S)$. So for each $r \in \mathbb{Q} \in [0, 1)$, define

$$N_r := ((N + r) \cup (N + r - 1)) \cap [0, 1)$$

Since $\mathcal{L}$ is closed under translations, unions, and intersections, we see that $N_r \in \mathcal{L}$. Furthermore, since $(N + r) \cap [0, 1)$ and $(N + r - 1) \cap [0, 1)$ are disjoint we have

$$\begin{aligned} m(N_r) &= m((N + r) \cap [0, 1)) + m((N + r - 1) \cap [0, 1)) \\ &= m(N \cap [-r, 1 - r)) + m(N \cap [1 - r, 2 - r)) \\ &= m(N \cap [-r, 2 - r)) = m(N). \end{aligned}$$

Thus each N_r has the same measure as N.

Now, we claim that

$$[0, 1) = \bigcup_{r \in \mathbb{Q} \cap [0,1)} N_r,$$

and that $N_r \cap N_s = \emptyset$ if $s \neq r$. Indeed, if $x \in [0, 1)$, then there exists $y \in N$ so that $x \in [y]$; that is, $x - y$ is some rational number r. Since $x, y \in [0, 1)$, we have either $r \in [0, 1)$ or $-r \in [0, 1)$. In the former case, we have $x = y + r \in (N + r) \cap [0, 1) \subset N_r$. In the latter case, let $s := r + 1 \in [0, 1)$ so that $x = y + r = y + s - 1 \in (N + s - 1) \cap [0, 1) \subset N_s$. Thus, $[0, 1) \subset \bigcup_{r \in \mathbb{Q} \cap [0,1)} N_r$, and the reverse inclusion is obvious.

Next, if $x \in N_r \cap N_s$ for $r, s \in \mathbb{Q} \cap [0, 1)$ we have $x \in [0, 1)$ and there exists $r' \in \{r, r-1\}$ and $s' \in \{s, s-1\}$ so that $x - r', x - s' \in N$. Since $(x - r') - (x - s') = s' - r' \in \mathbb{Q}$, we have $(x - r') \sim (x - s')$. Recall that N contains exactly one representative of each equivalence class, so it must be that $x - r' = x - s'$ and hence $r' = s'$. This implies either $r = s$, $r - 1 = s$, $r = s - 1$, or $r - 1 = s - 1$. The first and last cases both imply $r = s$ consequently $N_r = N_s$. The other two cases both say a non-negative number is equal to a negative number, which is a contradiction. We have shown that if $N_r \cap N_s$ is non-empty, then $N_r = N_s$.

Thus, we can write $[0, 1)$ as the disjoint union $\bigcup_{r \in \mathbb{Q} \cap [0,1)} N_r$, and since $\mathbb{Q} \cap [0, 1)$ is countable we have

$$1 = m([0, 1)) = \sum_{r \in \mathbb{Q} \cap [0,1)} m(N_r).$$

Recall that $m(N_r) = m(N)$ for each $r \in \mathbb{Q} \cap [0, 1)$. If $m(N) = 0$, the right-hand side above is zero, a contradiction. If $m(N) > 0$, the right-hand side is ∞, another contradiction. Thus, it must be that $N \notin \mathcal{L}$. $\square$

Since $\mathcal{B}_{\mathbb{R}} \subset \mathcal{L}$, the example in the previous proposition is also not Borel measurable. Another natural question is whether $\mathcal{L} \setminus \mathcal{B}_{\mathbb{R}}$ is non-empty. It turns out that there do exist subsets which are Lebesgue measurable but not Borel measurable, though the construction of such an example is quite tedious and involves defining a continuous function on the Cantor set (which importantly is an m-null set despite being uncountable).

2 Integrals

In class, we characterized the Riemann integrability of a function in terms of approximation by step functions. This characterization tells us that step functions are the "building blocks" of Riemann integrable functions. To define the Lebesgue integral, we will consider a generalization of step functions called "simple functions." A function will be Lebesgue integrable if it can be approximated by these simple functions in some appropriate way.

We also remark that though these notes are written with functions on $\mathbb{R}$ in mind, measure spaces offer much greater generality. Indeed, given an arbitrary measure space $(X, \mathcal{M}, \mu)$, one can define an integral for functions $f : X \to \mathbb{R}$ (or even $\mathbb{C}$-valued functions). In particular, one can make $\mathbb{R}^n$ into a measure space equipped with an n-dimensional Lebesgue measure for any $n \in \mathbb{N}$, and the corresponding integral recovers the iterated integral one studies in multi-variable calculus.

2.1 Simple functions

Definition 2.1. For $S \subset \mathbb{R}$, the **characteristic** (or **indicator**) **function of** S is the function $\chi_S \colon \mathbb{R} \to \mathbb{R}$ defined by

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}.$$

A function $\phi \colon \mathbb{R} \to \mathbb{R}$ is called a **simple function** if it is a finite linear combination of characteristic functions; that is, there exists $n \in \mathbb{N}$, numbers $a_1, \ldots, a_n \in \mathbb{R}$, and subsets $S_1, \ldots, S_n \subset \mathbb{R}$ such that $\phi = \sum_{i=1}^n a_i \chi_{S_i}$.

Observe that the numbers $a_1, \ldots, a_n \in \mathbb{R}$ and subsets $S_1, \ldots, S_n \subset \mathbb{R}$ need note be unique. For example,

$$1 \cdot \chi_{[0,1]} + 1 \cdot \chi_{[1/2,1]} = 1 \cdot \chi_{[0,1/2]} + 2 \cdot \chi_{[1/2,1]}.$$

But, if we insist that $a_1, \ldots, a_n$ are distinct and $S_i = \phi^{-1}(a_i)$, then this decomposition is unique. Moreover, $S_1, \ldots, S_n$ are *disjoint* subsets of $\mathbb{R}$. This is called the **standard representation of** ϕ, and in the above example is given by the right-hand side.

Example 2.2. For $a, b \in \mathbb{R}$ with $a < b$, let $f \colon [a,b] \to \mathbb{R}$ be a step function. That is, there exists a partition $a = x_0 < x_1 < \cdots < x_n = b$ and constants $c_1, \ldots, c_n \in \mathbb{R}$ such that $f(x) = c_i$ for all $x \in (x_{i-1}, x_i)$ and each $i = 1, \ldots, n$. Then f is equal to the following simple function

$$\sum_{i=1}^n c_i \chi_{(x_{i-1}, x_i)} + \sum_{j=0}^n f(x_j) \chi_{\{x_j\}}.$$

If we ignore the behavior of f on the partition points $\{x_0, x_1, \ldots, x_n\}$, then f agrees m-almost everywhere with the simple function

$$\sum_{i=1}^n c_i \chi_{(x_{i-1}, x_i)},$$

since the singleton sets $\{x_j\}$, $j = 0, \ldots, n$, are m-null sets.

2.2 Measurable functions

Next we must discuss what can be thought of as the measure space analogue of continuous functions. As we shall see, these form a very broad class of functions, and our discussion of Lebesgue integrability will be limited to this class.

Definition 2.3. A function $f \colon \mathbb{R} \to \mathbb{R}$ is said to be **Lebesgue** (resp. **Borel**) **measurable** if for every $S \in \mathcal{B}_{\mathbb{R}}$, $f^{-1}(S) \in \mathcal{L}$ (resp. $f^{-1}(S) \in \mathcal{B}_{\mathbb{R}}$).

Since $\mathcal{B}_{\mathbb{R}} \subset \mathcal{L}$, all Borel measurable functions are Lebesgue measurable. It follows from simple set theoretic observations that any continuous function $f \colon \mathbb{R} \to \mathbb{R}$ is Borel (and hence Lebesgue) measurable. For a characteristic function χ_S or a subset $S \subset \mathbb{R}$, note that for $T \subset \mathbb{R}$ we have

$$\chi_S^{-1}(T) = \begin{cases} \mathbb{R} & \text{if } 0, 1 \in T \\ S & \text{if } 1 \in T\text{, but } 0 \notin T \\ S^c & \text{if } 0 \in T\text{, but } 1 \notin T \\ \emptyset & \text{otherwise} \end{cases}.$$

Consequently, χ_S is Lebesgue (resp. Borel) measurable if and only if $S \in \mathcal{L}$ (resp. $S \in \mathcal{B}_{\mathbb{R}}$). Similarly, a simple function $\phi = \sum_{i=1}^n a_n \chi_{S_i}$ is Lebesgue (resp. Borel) measurable if and only if $S_1, \ldots, S_n \in \mathcal{L}$ (resp. $S_1, \ldots, S_n \in \mathcal{B}_{\mathbb{R}}$).

The following proposition indicates that the set of measurable functions is closed under the usual arithmetic and limit operations.

Proposition 2.4. *Let $f, g \colon \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable functions. Then $f + g$, $f - g$, $f \cdot g$, and f/g are measurable. If $(f_n)_{n \in \mathbb{N}}$ is a sequence of Lebesgue measurable functions then $\limsup_{n\to\infty} f_n$ and $\liminf_{n\to\infty} f_n$ are Lebesgue measurable. In particular, if $(f_n)_{n \in \mathbb{N}}$ converges pointwise to f, then f is Lebesgue measurable.*

2.3 The Lebesgue integral

To begin our discussion of the Lebesgue integral, we must work with *positive* Lebesgue measurable functions. We let L^+ denote the set of Lebesgue measurable functions $f: \mathbb{R} \to [0,\infty]$. Note that we do allow functions to take on the value of ∞, though we will have to exercise a little bit of caution. So long as a function is finite m-almost everywhere, we will be able to work with it.

Definition 2.5. For a simple function $\phi \in L^+$ with standard representation $\sum_{i=1}^n a_i \chi_{S_i}$ (note that $a_1, \ldots, a_n \in [0,\infty]$), its **Lebesgue integral** is defined as the (possibly infinite) quantity

$$\int_{\mathbb{R}} \phi \; dm := \sum_{i=1}^n a_i m(S_i).$$

Example 2.6. As shown in Example 2.2, step functions are simple functions. Let $c_1, \ldots, c_n \in [0,\infty)$ and let $a = x_0 < x_1 < \cdots < x_n = b$ be a partition. Let $f: [a,b] \to [0,\infty]$ be a step function taking the value c_i on the interval (x_{i-1}, x_i), for $i = 1, \ldots, n$. Then its Lebesgue integral of f is

$$\begin{aligned} \int_{\mathbb{R}} f \; dm &= \sum_{i=1}^n c_i m((x_{i-1}, x_i)) + \sum_{j=0}^n f(x_j) m(\{x_j\}) \\ &= \sum_{i=1}^n c_i (x_i - x_{i-1}). \end{aligned}$$

Thus the Lebesgue integral of f agrees with its Riemann integral. We will see that this holds in general (*cf.* Theorem 2.12).

Example 2.7. Consider the simple function $\chi_{\mathbb{Q}\cap[0,1]}$. Since $\mathbb{Q} \cap [0,1] \in \mathcal{L}$, we have $\chi_{\mathbb{Q}\cap[0,1]} \in L^+$ and hence its Lebesgue integral is given by the above definition:

$$\int_{\mathbb{R}} \chi_{\mathbb{Q}\cap[0,1]} \; dm = m(\mathbb{Q} \cap [0,1]) = 0.$$

Recall that we showed this same function is **not** Riemann integrable.

Definition 2.8. For a function $f \in L^+$, its **Lebesgue integral** is defined as the (possibly infinite) quantity

$$\int_{\mathbb{R}} f \; dm := \sup \left\{ \int_{\mathbb{R}} \phi \; dm \colon \phi \le f, \;\; \phi \in L^+ \text{ simple} \right\}$$

Given a Lebesgue measurable function $f: \mathbb{R} \to \mathbb{R}$, consider the disjoint subsets

$$\begin{aligned} S_+ &:= f^{-1}([0,\infty)) \\ S_- &:= f^{-1}((-\infty, 0)). \end{aligned}$$

Since the intervals $[0,\infty)$ and $(-\infty,0)$ are Borel measurable, $S_+, S_- \in \mathcal{L}$ by virtue of f being Lebesgue measurable. Hence by Proposition 2.4, the functions $f_+ := f \cdot \chi_{S_+}$ and $f_- := -f \cdot \chi_{S_-}$ are Lebesgue measurable. These functions are called the **positive** and **negative parts of** f, respectively. By definition, we have $f = f_+ - f_-$ and $f_+, f_- \in L^+$. Also note that $|f| = f_+ + f_-$.

Definition 2.9. Let $f: \mathbb{R} \to \mathbb{R}$ be a measurable function. If one of $\int_{\mathbb{R}} f_+ \; dm$ or $\int_{\mathbb{R}} f_- \; dm$ is finite, we define the **Lebesgue integral of** f to be the quantity:

$$\int_{\mathbb{R}} f \; dm = \int_{\mathbb{R}} f_+ \; dm - \int_{\mathbb{R}} f_- \; dm.$$

We say f is **Lebesgue integrable** if both $\int_{\mathbb{R}} f_+ \; dm$ and $\int_{\mathbb{R}} f_- \; dm$ are finite. Equivalently, since $|f| = f_+ + f_-$, f is Lebesgue integrable if $\int_{\mathbb{R}} |f| \; dm$ is finite, in which case $\left| \int_{\mathbb{R}} f \; dm \right| \le \int_{\mathbb{R}} |f| \; dm$ by the triangle inequality.

As indicated by the following proposition, the Lebesgue integral satisfies the same properties we have come to expect with the Riemann integral.

Proposition 2.10. *Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be Lebesgue integrable functions and $a \in \mathbb{R}$. Then $f + ag$ is Lebesgue integrable with*

$$\int_{\mathbb{R}} f + ag\ dm = \int_{\mathbb{R}} f\ dm + a \int_{\mathbb{R}} g\ dm.$$

If $f \leq g$, then

$$\int_{\mathbb{R}} f\ dm \leq \int_{\mathbb{R}} g\ dm.$$

Next we observe that integral is not affected by the behavior on m-null sets.

Proposition 2.11. *Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is zero m-almost everywhere. Then f is Lebesgue integrable with $\int_{\mathbb{R}} f\ dm = 0$. Consequently, if $g\colon \mathbb{R} \to \mathbb{R}$ is such that $f = g$ m-almost everywhere, then $\int_{\mathbb{R}} f\ dm = \int_{\mathbb{R}} g\ dm$.*

Proof. First note that f is a Lebesgue measurable function. Indeed, let $N = f^{-1}(\mathbb{R} \setminus \{0\})$ which is an m-null set by assumption. Since $(\mathbb{R}, \mathcal{L}, m)$ is complete, any subset of N is Lebesgue measurable. Then for $S \subset \mathbb{R}$, $f^{-1}(S)$ is the union N^c (if $0 \in S$) and a subset of N, and hence is Lebesgue measurable. Thus, f is a Lebesgue measurable function.

Now, since f is Lebesgue integrable if and only if $|f|$ is, we may assume $f \geq 0$. Let $\phi \in L^+$ be any simple function such that $\phi \leq f$, say with standard representation $\phi = \sum_{i=1}^n a_n \chi_{S_i}$. Without loss of generality, we may assume $a_i > 0$ for each $i = 1, \ldots, n$. Then since $\phi \leq f$, we must have $S_i \subset N$ for each $i = 1, \ldots, n$, and consequently $m(S_i) = 0$. Hence

$$\int_{\mathbb{R}} \phi\ dm = \sum_{i=1}^n a_i m(S_i) = 0.$$

Consequently, $\int_{\mathbb{R}} f\ dm$, as the supremum of such integrals, is zero.

The final statement follows from considering $\int_{\mathbb{R}} f - g\ dm$ and using Proposition 2.10. □

Finally, we conclude with the following theorem showing that the Riemann integral is subsumed by the Lebesgue integral.

Theorem 2.12. *For $a, b \in \mathbb{R}$ with $a < b$, let $f\colon [a, b] \to \mathbb{R}$ be a bounded function.*

a. If f is Riemann integrable, then f is Lebesgue measurable and

$$\int_a^b f(x)\ dx = \int_{\mathbb{R}} f \cdot \chi_{[a,b]}\ dm.$$

*b. f is Riemann integrable if and only if $\{x \in [a, b] \colon f \text{ is discontinuous at } x\}$ is an m-null set (**Lebesgue's integrability condition**).*

References

[1] Gerald B. Folland; *Real Analysis*, Second Edition, John Wiley & Sons, Inc., New York. (1999).