Finitely Summable K-homology for Crossed Products

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K-homology

- A dual (genealized homology) theory to K-theory generalizing topological K-homology
- A contravariant functor from separable C*-algebras to $\mathbb{Z}/2\mathbb{Z}$ graded abelian groups.
- **Kasparov (analytic) picture**: K-homology is built out of "generalized Fredholm (elliptic) operators," called **Fredholm Modules**

Fredholm Modules

Definition

Let A be a C*-algebra. An (ungraded) **Fredholm module** over A is a triple (H, ρ, F) where:

- H is a separable Hilbert space
- $\rho : A \to B(H)$ is a representation
- $F \in B(H)$ satisfies:
  1. $(F^2 - 1)\rho(a) \in K(H)$ for each $a \in A$
  2. $(F - F^*)\rho(A) \in K(H)$ for each $a \in A$
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- A *graded* Fredholm module is a Fredholm module equipped with a $\mathbb{Z}/2\mathbb{Z}$ grading for $H := H^+ \oplus H^-$, where ρ is a representation by even (grading-preserving) operators, and F is an odd (grading-reversing) operator.

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- Ungraded Fredholm modules generate $K^1(A)$, graded Fredholm modules generate $K^0(A)$.

K-homology pairs with K-theory

Definition

Let (H, ρ, F) be an ungraded Fredholm module over A, u a unitary in $M_k(A)$, $P_k = 1 \otimes \frac{1+F}{2} \in B(\mathbb{C}^k \otimes H)$, $(H, \rho, \begin{bmatrix} 0 & V \\ U & 0 \end{bmatrix})$ a graded Fredholm module over A, and p a projection in $M_k(A)$. Then:

$$\langle [u], [H, \rho, F] \rangle := \text{Fred-Index}(P_k(1 \otimes \rho)uP_k - (1 - P_k))$$

gives a bilinear pairing:

$$\text{Index} : K_1(A) \times K^1(A) \to \mathbb{Z} \text{ and}$$

$$\langle [p], [H, \rho, \begin{bmatrix} 0 & V \\ U & 0 \end{bmatrix}] \rangle := \text{Fred-Index}((1 \otimes \rho)(p)(1 \otimes U)(1 \otimes \rho)(p))$$

gives a bilinear pairing:

$$\text{Index} : K_0(A) \times K^0(A) \to \mathbb{Z}$$

Schatten Classes

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Let H be separable Hilbert space and $T \in \mathcal{K}(H)$. The $n-$th **singular value**, $s_n(T)$, of T is the n-th eigenvalue of $|T| = (T^*T)^{\frac{1}{2}}$ when ordered from largest to smallest.

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For example:

- $\mathcal{L}^1(H)$ - Trace Class Operators
- $\mathcal{L}^2(H)$ - Hilbert-Schmidt Operators

Summability

Definition

A Fredholm module (H, ρ, F) is **p-summable** over A if there is a dense *-subalgebra $\mathcal{A} \subseteq A$ such that, for all $a \in \mathcal{A}$:

$$(F - F^*)\rho(a) \in \mathcal{L}^{2p}(H) \ (F^2 - 1)\rho(a), \in \mathcal{L}^{2p}(H), \quad \text{and } [F, \rho(a)] \in \mathcal{L}^p(H).$$

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- If a class in $K^*(A)$ can be represented by a finitely summable representative, cyclic cohomology gives us a more computible formula for the index pairing.
- (Rave) For any C*-algebra A, any odd Fredholm module that is finitely summable on *all* of A represents [0] in $K^1(A)$.

The Unbounded Picture

Definition (Unbounded Fredholm module)

Let A be a C*-algebra. An **Unbounded Fredholm module** (H, ρ, D) over A is a triple (H, ρ, D) where the operator F is replaced by a suitable unbounded self-adjoint operator D. (A graded version exists as well)

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Proposition (Baaj,Julg)

The bounded transform maps a p-summable unbounded cycle to a cycle that is p-summable on a suitable subalgebra

Bounded and unbounded

Proposition (Kasparov, Connes, Folklore)

Let M be smooth compact $n-$manifold. Then every class in $K^*(C(M))$ can be represented by an unbounded cycle that is p-summable on $C^\infty(M)$, for every $p > n$.

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- This reflects that an unbounded cycle encodes the geometry of the manifold, while the bounded cycle only encodes the "conformal geometry" and the notion of length is lost.

Uniform Summability

Definition

A C*-algebra A has **uniformly p-summable** K-homology if there is a dense *-subalgebra $\mathcal{A} \subseteq A$ such that every class in $K^*(A)$ can be represented by a Fredholm module that is p-summable on $\mathcal{A}$.

Known Examples of Uniform Summability

Known Noncommutative Examples:

C^*-Algebra	subalgebra	$p >$	Auts.
AF algebras: $A = \overline{\cup_n A_n}$	$\cup_n A_n$	0	Rave
A_θ	$\sum a_{m,n} U^m V^n$ $a_{m,n} \in S(\mathbb{Z}^2)$	2	Connes
Γ hyperbolic: $C(\partial\Gamma) \rtimes \Gamma$	$\mathrm{Lip}(\partial\Gamma, d) \rtimes_{\mathrm{alg}} \Gamma$	$\dim_H(\partial\Gamma, d)$	Emerson Nica
O_A	$\mathbb{C}^*[S_1, S_2, \ldots, S_n]$	0	Goffeng Mesland
Ruelle Algebras: $S_\varphi(U_\varphi) \rtimes_\alpha \mathbb{Z}$	$\mathrm{Lip}_c(G_s(G_u)) \rtimes_{\alpha,\mathrm{alg.}} \mathbb{Z}$	$C \cdot h(\varphi)$	Gerontogiannis

Example failing uniform summability

Theorem (Goffeng-Mesland)

There is a class in $K^1(\bigoplus_{n\in\mathbb{N}} C(S^{2n-1}))$ that does not admit a finitely summable representative on any dense subalgebra.

Generalizing

- The quintessential example of a noncommutative manifold is **the irrational rotation algebra**, $A_\theta \cong C(\mathbf{S}^1) \rtimes \mathbb{Z}$.
- $A_\theta \cong C(\mathbf{S}^1) \rtimes \mathbb{Z}$ was shown to have uniformly $p > 2$ summable K-homology
- I am interested in generalizing this result to other crossed products, by generalizing both the space being acted on and the group acting on it.

Generalizing the group

(In Progress)

Let Γ be a countable discrete group that satisfies the strong Baum-Connes Conjecture (there exists a γ element and it equals 1) and such that $\underline{E}\Gamma$ can be chosen to be a Γ-proper, Γ-cocompact manifold. Then if M is a smooth, compact manifold and Γ acts on M via diffeomorphisms, then all the K-homology classes of $C(M) \rtimes \Gamma$ can be represented by finitely summable Fredholm modules with a uniform bound on the degree of summability.

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- Another class of examples in this framework include when Γ is the fundamental group of a flat manifold

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Let X be a Cantor set and $\varphi : X \to X$ a homeomorphism inducing the automorphism $\alpha : C(X) \to C(X)$. Then, every class in $K^1(C(X) \rtimes_\alpha \mathbb{Z})$ can be represented by an unbounded finitely summable cycle on $\mathrm{Lip}(X) \rtimes_{alg} \mathbb{Z}$, and the degree of summability can be chosen arbitrarily small but greater than 1.

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- Because we are working with a $\mathbb{Z}$ action we can utilize the Pimsner-Voiculescu sequence. We also benefit from the Cantor Set having nice K-theory (and thus K-homology)
- K^0 seems to depend more on the action.

Further Directions

- (In progress) If X is a finite CW-complex of dimension n, then $C(X)$ has uniformly $p > n$ summable K-homology by unbounded cycles on the algebra of functions whose restriction to each cell is smooth.

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- The name of the game is finding nice representatives for KK elements (i.e. we know how to represent the Kasparov product, the Kasparov product produces finitely summable classes, and we know what happens with the subalgebra)
- The unbounded picture is often helpful, but, as we have seen, it is limiting

Bibliography

- Connes, Alain. *Noncommutative Geometry*. (1994)
- Connes, Alain. "Compact metric spaces, Fredholm modules, and hyperfiniteness." (1989).
- Emerson, Heath and Bogdan Nica. "Finitely summable Fredholm modules for boundary actions of hyperbolic groups."(2012).
- Gerontogiannis, Dimitris. "On finitely summable Fredholm modules from Smale spaces." (2021).
- Goffeng, Magnus and Bram Mesland. "Finite Summability and Spectral Triples for Cuntz-Krieger Algebras." (2014).
- Higson, Nigel and Roe, John. *Analytic K-homology*. (2000).
- Hadfield, Tom. "K-homology of certain group C^*-algebras." (2002).
- Kasparov, G.G. "Topological Invariants of Elliptic Operators I: K-homology."(1975).
- Puschnigg, Michael. "Finitely summable Fredholm modules over higher rank groups and lattices."(2008).
- Rave, Stephen. "On Finitely Summable K-homology." (2012).

Questions?

Thank you!!