A Low-Dimensional Counterexample to the HK-Conjecture

Rachel Chaiser[1]

University of Colorado Boulder

August 17, 2023

[1]Partially supported by NSF grant DMS 2000057.

The HK-Conjecture

Conjecture (Matui)

Let $\mathcal{G}$ be a groupoid that is 2nd countable, locally compact Hausdorf, étale, essentially principal, minimal, and ample. Then

$$K_*(C_r^*(\mathcal{G})) \cong \bigoplus_i H_{2i+*}(\mathcal{G}).$$

Some results:

Positive	Negative
AF groupoids [M '15, FKPS '19]	Certain generalized odometers [S '20, D '23]
Groupoids from shifts of finite type [M '15]	Certain non-free actions on the Cantor set [OS '21]
Certain k-graph groupoids [FKPS '19]	
Dynamic asymptotic dimension $\leq$ 2 [BDGW '23]	

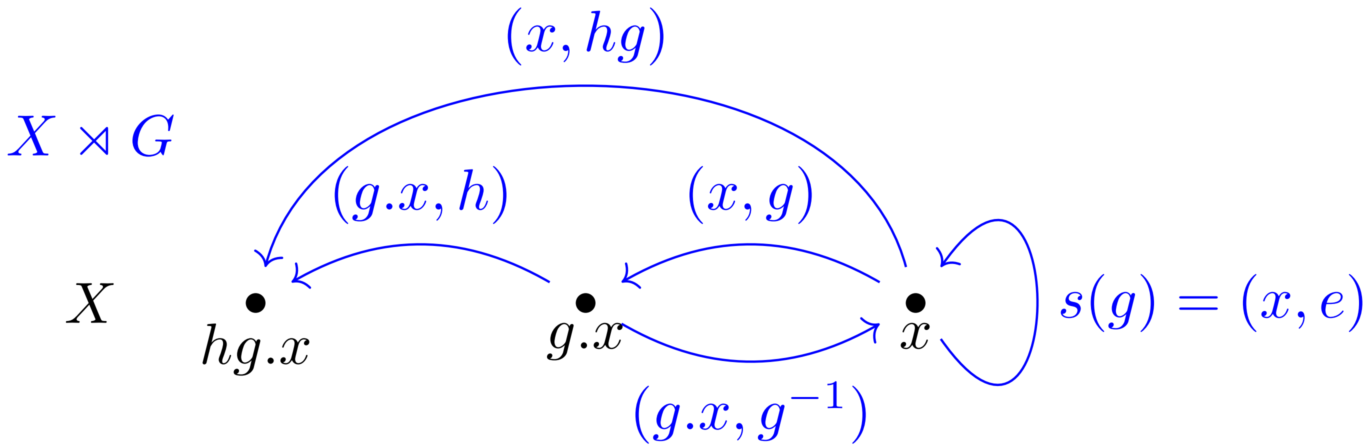
principal

principal

ess principal

Action Groupoids

Let X be a space and G a (topological) group acting on X
The action groupoid is $X \rtimes G := X \times G$ with compositon encoding group multiplication



$X \rtimes G$ is:

- étale $\iff G$ is discrete
- principal and minimal $\iff G \curvearrowright X$ freely and minimally (resp.)
- ample $\iff X$ is a Cantor set

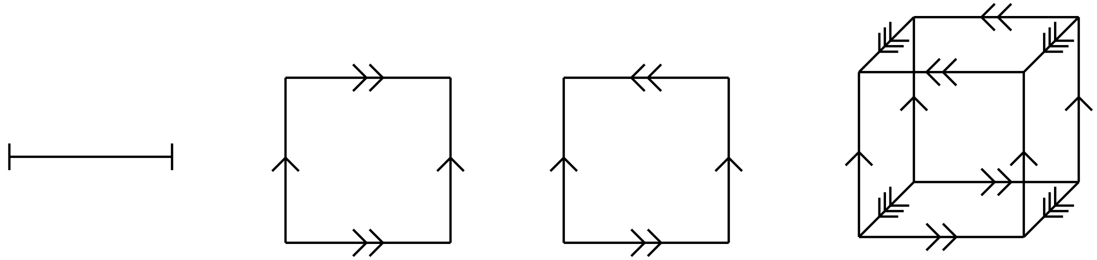
Flat Manifolds

Definition

A flat manifold Y is a closed Riemannian d-manifold with zero curvature.

- $Y \cong \dfrac{\mathbb{R}^d}{\pi_1(Y)}$ and $\pi_1(Y)$ is a torsionfree subgroup of $O(n) \ltimes \mathbb{R}^d$

Examples:



S^1 Torus Klein bottle $S^1 \times K$

Theorem (Epstein–Shub '68)

Let Y be a flat manifold. Then there exists a locally expansive surjection $g : Y \to Y$. Moreover, g is an n-fold cover for some $n \geq 2$.

Generalized Odometers

Let Y be a flat manifold and $g : Y \to Y$ an expansive self-cover.

Construct a Cantor set $\Omega = \varprojlim \left(\frac{\pi_1(Y)}{g_*^i \pi_1(Y)}, \text{ coset inclusion} \right)$.

Then $\pi_1(Y) \curvearrowright \Omega$ by group multiplication.

The action groupoid $\mathcal{G} = \Omega \rtimes \pi_1(Y)$ is a generalized odometer.

Example ($Y = S^1 \subseteq \mathbb{C}$ and $g : Y \to Y$ is $g(z) = z^2$)

$$\frac{\mathbb{Z}}{2\mathbb{Z}} \xleftarrow{\times 2} \frac{\mathbb{Z}}{4\mathbb{Z}} \xleftarrow{\times 2} \frac{\mathbb{Z}}{8\mathbb{Z}} \xleftarrow{\times 2} \cdots$$

In binary: $\frac{\mathbb{Z}}{2\mathbb{Z}} = \{0, 1\}$, $\frac{\mathbb{Z}}{4\mathbb{Z}} = \{00, 01, 10, 11\}$, $\ldots$

Then $\Omega = \{0, 1\}^{\mathbb{N}}$ and $\mathbb{Z} \curvearrowright \Omega$ as the $(0, 1)$-odometer:

$$1 \cdot (0, 0, 0, 0 \ldots) = (1, 0, 0, 0, \ldots)$$
$$1 \cdot (1, 0, 0, 0 \ldots) = (0, 1, 0, 0, \ldots)$$
$$1 \cdot (1, 1, 0, 0 \ldots) = (0, 0, 1, 0, \ldots)$$

More on Generalized Odometers

Let Y be a flat manifold and $g : Y \to Y$ an n-fold expansive self-cover.
$\mathcal{G} = \Omega \rtimes \pi_1(Y)$ satisfies the hypotheses of the HK-conjecture:

1. étale: $\pi_1(Y)$ discrete
2. ample: Ω is a Cantor set
3. minimal: action is minimal [Cortez–Petite '08]
4. principal: g expansive $\implies$ action is free [Deeley '23]

Theorem (Scarparo '20, Deeley '23)

$$H_*(\mathcal{G}) \cong \varinjlim(H_*(Y), \mathrm{tr}_H)$$

where tr_H *is the transfer map in homology, which satisfies* $g_* \circ \mathrm{tr}_H = \times n$.

$$K_*(C_r^*(\mathcal{G})) \cong \varinjlim(K_*(Y), \mathrm{tr}_K)$$

where tr_K *is the transfer map in* K*-homology.*

The Goal

Let Y be a flat manifold, $g : Y \to Y$ an n-fold expansive self-cover, and $\mathcal{G} = \Omega \rtimes \pi_1(Y)$ the associated generalized odometer.

Theorem (Scarparo '20, Deeley '23)

$$H_*(\mathcal{G}) \cong \varinjlim (H_*(Y), \mathrm{tr}_H)$$

where tr_H *is the transfer map in homology, which satisfies* $g_* \circ \mathrm{tr}_H = \times n$.

$$K_*(C_r^*(\mathcal{G})) \cong \varinjlim (K_*(Y), \mathrm{tr}_K)$$

where tr_K *is the transfer map in* K*-homology.*

Goal: Find a flat manifold Y with

$$K_*(Y) \ncong \bigoplus_i H_{2i+*}(Y)$$

The Atiyah–Hirzebruch Spectral Sequence

For a CW-complex X "$\bigoplus_i H^{2i+*}(X)$ approximates $K^*(X)$":

$$E_2^{p,q} = H^p(X; K^q(\mathrm{pt})) \text{ converges to } K^{p+q}(X)$$

$H^0(X)$	$H^1(X)$	$H^2(X)$	$H^3(X)$	$H^4(X)$	$\cdots$
0	0	0	0	0	$\cdots$
$H^0(X)$	$H^1(X)$	$H^2(X)$	$H^3(X)$	$H^4(X)$	$\cdots$
0	0	0	0	0	$\cdots$
$H^0(X)$	$H^1(X)$	$H^2(X)$	$H^3(X)$	$H^4(X)$	$\cdots$

Differentials d_r $\xrightarrow{r}$ $\downarrow_{r-1}$ on rth page.

Taking d_r-cohomology gives $(r+1)$th page, and better approximation of $K^*(X)$.

Counterexamples with $\dim(Y) \geq 9$

Theorem (Deeley '23)

For each $d \geq 9$ there exists a flat manifold Y with $\dim(Y) = d$ and an expansive self-cover $g : Y \to Y$ such that the associated generalized odometer is a counterexample to the HK-conjecture.

- $d_3 \neq 0$

Question: What is the minimal dimension of a flat manifold giving a counterexample to the HK-conjecture?

More on the Atiyah–Hirzebruch Spectral Sequence

Free

$$
\begin{array}{cccccc}
H^0(X) & H^1(X) & H^2(X) & H^3(X) & H^4(X) & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots \\
H^0(X) & H^1(X) & H^2(X) & H^3(X) & H^4(X) & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots \\
H^0(X) & H^1(X) & H^2(X) & H^3(X) & H^4(X) & \cdots
\end{array}
$$

d_3

(*)

Facts:

- If $\dim(X) \leq 3$ then $K^*(X) \cong \bigoplus_i H^{2i+*}(X)$
- (*) d_r are 'pure torsion morphisms' $\implies d_r = 0$ for all r if $\dim(X) \leq 4$
- There is a short exact sequence when $\dim(X) \leq 4$

$$0 \to H^2(X) \to \widetilde{K}^0(X) \to H^4(X) \to 0$$

AHSS Example: $\mathbb{R}P^4$

Let $X = \mathbb{R}P^4$. The AHSS becomes

$$\begin{array}{cccccccc}
\mathbb{Z} & 0 & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\mathbb{Z} & 0 & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\mathbb{Z} & 0 & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & 0 & 0 & \cdots
\end{array}$$

There is a short exact sequence

$\mathbb{Z}_2 \oplus \mathbb{Z}_2$ $\mathbb{Z}_4$

$$0 \to \mathbb{Z}_2 \to \widetilde{K}^0(\mathbb{R}P^4) \to \mathbb{Z}_2 \to 0$$

and $\widetilde{K}^0(\mathbb{R}P^4) \cong \mathbb{Z}_4$.

But $\mathbb{R}P^4$ is not flat.

Counterexample with $\dim(Y) = 4$

Theorem (C.)

There exists a flat manifold Y with $\dim(Y) = 4$ and an expansive self-cover $g : Y \to Y$ such that the associated generalized odometer is a counterexample to the HK-conjecture.

Proof sketch:

dad(G) = dim(Y) = 4

- Show $\mathrm{Torsion}\left(\bigoplus_i H^{2i}(Y)\right) \cong \mathbb{Z}_2^4$
- Use characteristic classes and ring structure of $H^*(Y;\mathbb{Z}_2)$[1] to find an explicit line bundle η on Y where $\eta - 1 \in K^0(Y)$ has order 4
- This gives $\mathrm{Torsion}(K^0(Y)) \cong \mathbb{Z}_2^2 \oplus \mathbb{Z}_4$
- Use UCT to convert to (K-)homology
- Show torsion subgroups in $\varinjlim(-, \mathrm{tr})$ still differ[2]

[1] Ring structure is computed in [KM]
[2] See [D], [ES]

References

[1] C. Bönicke, C. Dell'Aiera, J. Gabe, and R. Willett. *Dynamic asymptotic dimension and Matui's HK conjecture*. Proc. Lond. Math. Soc., (2023).

[2] M. I. Cortez and S. Petite. *G-odometers and their almost one-to-one extensions*. J. Lond. Math. Soc., (2008).

[3] R. Deeley. *A counterexample to the HK-conjecture that is principal*. Ergodic Theory Dynam. Systems, (2023).

[4] D. Epstein and M. Shub. *Expanding endomorphisms of flat manifolds*. Topology, (1968).

[5] C. Farsi, A. Kumjian, D. Pask, and A. Sims. *Ample groupoids: equivalence, homology, and Matui's HK conjecture*. Münster J. Math., (2019).

[6] Y. Kamishima and M. Masuda. *Cohomological rigidity of real Bott manifolds*. Algebr. Geom. Topol., (2009).

[7] E. Ortega and E. Scarparo. *Almost finiteness and homology of certain non-free actions*. arXiv:2007.02329, (2020).

[8] E. Scarparo. *Homology of odometers*. Ergodic Theory Dynam. Systems, (2020).