Almost commuting matrices and Bott periodicity

Plan:

1. Almost commuting matrices

2. K-theory and Bott periodicity

3. Representation theory

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1. Almost commuting matrices

Motivating question:

"Ulam stability"

"Is (an almost solution) (almost a solution)?"

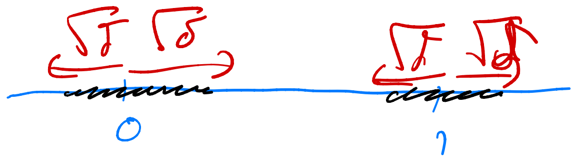
"Is an almost solution almost a solution?"

Sample question:

$\|x^2 - x\| < \delta \leq \frac{1}{4}$

If $x \in A_{sa}$ satisfies $x^2 - x \simeq 0$, must x be close to a projection?

Yes!



$f = \chi_{(1/2,\infty)}$

$\|f(x) - x\| \leq \sqrt{\delta}$

More precise answer:

$\forall \varepsilon > 0\ \exists \delta > 0$ such that if $x \in A_{sa}$ is a "δ-projection" $\|x^2 - x\| < \delta$ then x is "ε-close" $\|x - p\| < \varepsilon$ to an honest projection.

Is the following true? Halmos '68

"$\forall \varepsilon > 0 \; \exists \delta > 0$ s.t. if $u, v \in \mathcal{U}_n$ ← $n \times n$ unitaries

$\|uv - vu\| < \delta \iff \|1 - u^{-1}v^{-1}uv\| < \delta$

δ-commute, then u, v are ε-close

to a commuting pair"

Answer (Voiculescu '83): No!

Why? (Exel - Loring '89):

For $u, v \in \mathcal{U}_n$ with $\|1 - u^{-1}v^{-1}uv\| \approx 0$

define

$$\boxed{\omega(u,v) = \underset{t \in [0,1]}{\text{wind-}\#}\Big(\det\big((1-t) + t u^{-1}v^{-1}uv\big)\Big) \in \mathbb{Z}}$$

$$\omega(u,v) = \text{wind-}\#\left(\det\left((1-t)+t\,u^{-1}v^{-1}uv\right)\right)$$

$$(u')^{-1}(v')^{-1}u'v'$$

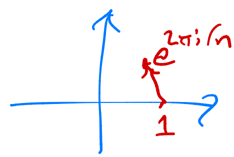
$$t \in [0,1]$$

Fact: if $\omega(u,v) \neq 0$, u, v not ε-close to a commuting pair. ($\varepsilon \sim 0.4$)

shift   mult.

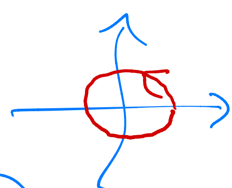
Example: $b_n = \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ 0 & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix}$, $v_n = \begin{pmatrix} 1 & & & & \\ & e^{2\pi i \frac{1}{n}} & & & \\ & & e^{2\pi i \frac{2}{n}} & & \\ & & & \ddots & \\ & & & & e^{2\pi i \frac{n-1}{n}} \end{pmatrix}$

$$b_n^{-1} v_n^{-1} b_n v_n = e^{2\pi i/n}$$



So: $\omega(b_n, v_n) = \text{wind}\#\ \det\left((1-t)+t e^{2\pi i/n}\right)$

$$= \text{wind}\#\ \left((1-t)+t e^{2\pi i/n}\right)^n$$



$$= 1$$

② K-theory and Bott periodicity

A : unital C^*-algebra

Bott periodicity theorem ('59, '65):

Bott ↓ Wood ↙

$$\boxed{K_1(C(S^1,A)) \cong K_0(A)}$$

Proposition (Atiyah '67):

To prove Bott periodicity, sufficient to construct for each A a natural map

$$\alpha_A : K_1(C(S^1,A)) \longrightarrow K_0(A)$$

$$\alpha_{\mathbb{C}} : K_1(C(S^1)) \longrightarrow K_0(\mathbb{C}) \cong \mathbb{Z}$$

such that $\alpha_{\mathbb{C}}(b) = 1$

↑ $b(z) = \bar{z}$

Construction of α_A:

Define $v_t \in B(L^2(S^1))$, $t \in [1,\infty)$ $\cong \ell^2(\mathbb{Z})$

$$v_t : \delta_n \longmapsto \begin{cases} e^{2\pi i \frac{n}{t}} \delta_n , & 0 \le n \le t \\ \delta_n , & \text{o/w} \end{cases}$$

Represent $C(S^1, A) = C(S^1) \otimes A$ on $L^2(S^1) \otimes H = L^2(S^1, H)$ and define

faithful rep.n' on A

$$\alpha_A : K_1(C(S^1,A)) \longrightarrow K_0(A \otimes \mathcal{K}) \cong K_0(A)$$

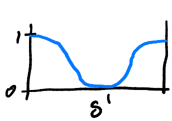
$$[u] \longmapsto [e(u, v_t \otimes 1)] - \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$$

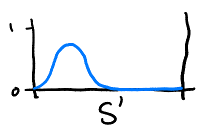
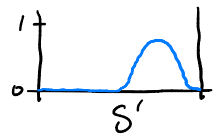
$t \gg 0$

almost a projection

Here

$$e(u,v) = \begin{pmatrix} f(v) & g(v) + h(v)\cdot u \\ u^* h(v) + g(v) & 1 - f(v) \end{pmatrix}$$

$f =$ 

$g =$ , $h =$ 

Compute $\alpha_{\mathbb{C}}(b) = [e(b, v_t)] - \left[\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right]$:

id on δ_n, $n \le 0$

$$v_t \approx \begin{pmatrix} \ddots & & & & & & \\ & 1 & & & & & \\ & & e^{2\pi i \frac{1}{t}} & & & & \\ & & & e^{2\pi i \frac{2}{t}} & & & \\ & & & & \ddots & & \\ & & & & & e^{2\pi i \frac{\lfloor t \rfloor}{t}} & \\ & & & & & & 1 \\ & & & & & & & \ddots \end{pmatrix}$$

id on δ_n, $n > t$

$$b = \begin{pmatrix} \ddots & & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & \ddots \\ & & & & \ddots & 1 \\ & & & & & 0 & \ddots \end{pmatrix}$$

Theorem (Exel–Loring '91):

If $u, v \in \mathcal{U}_n$, $u^{-1}v^{-1}uv \approx 1$, then:

$$\text{wind-}\#\left(\det\left((1-t) + t\, u^{-1}v^{-1}uv\right)\right) = 1$$

$$\parallel$$

$$[e(u, v)] - \left[\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right] = \alpha_{\mathbb{C}}(b)$$

A closer look at this theorem: Atiyah–Singer '62 for S^1

for $u \in C^\infty(S^1)$ and $n \gg 0$

$$\operatorname{wind-\#}\left(\det\left((\mathbb{1}-t) + t u^{-1} v_n^{-1} u v_n\right)\right) \in \mathbb{Z}$$

$$= \int_{S^1 \times \mathbb{R} = TS^1} \operatorname{ch}(E_u)$$

$$\| $$

$$\underbrace{[e(u, v_n)] - \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]}_{= \left\langle \frac{i\,d}{d\theta},\; u \right\rangle} \in \mathbb{Z}$$

$\frac{i\,d}{d\theta} \in K_1(C(S^1))$, $u \in K^1(C(S^1))$

③ Representation theory

Note:

$$\left(\begin{matrix} u, v \in U_n \text{ with} \\ u^{-1}v^{-1}uv = 1 \end{matrix}\right) \longleftrightarrow \left(\begin{matrix} \text{unitary rep.ns} \\ \pi : \mathbb{Z}^2 \to U_n \end{matrix}\right)$$

Similarly:

"approximate rep.n"

$$\left(\begin{matrix} u, v \in U_n \text{ with} \\ u^{-1}v^{-1}uv \approx 1 \end{matrix}\right) \longleftrightarrow \left(\begin{matrix} \text{quasi-rep.ns} \\ \mathbb{Z}^2 \to U_n \end{matrix}\right)$$

"Voiculescu's theorem":

There are quasi-representations of $\mathbb{Z}^2 = \langle s, t \mid s^{-1}t^{-1}st \rangle$ that are not close to actual representations.

Are all quasi-rep.ns close to rep.ns for

1) $G_1 = \langle s, t \mid s^{-1}tst \rangle$?

$\uparrow$

$\pi_1\Big($  $\Big)$

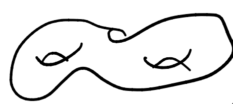
Yes!

Eilers–Pedersen–Loring '98

Eilers–Shulman–Sørensen '18

2) $G_2 = \langle s, t, u, v \mid (s^{-1}t^{-1}st)(u^{-1}v^{-1}uv) \rangle$?

$\uparrow$

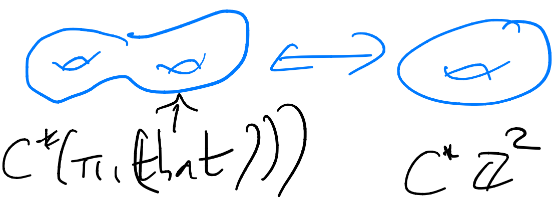
$\pi_1\Big($  $\Big)$

No!

Question:

$\mathbb{Z}^2$

Say $\pi : \pi_1($  $) \longrightarrow U_n$ is a quasi-rep.n with $\omega(\pi) = 0$. Is π close to an actual rep.n?

(Yes if )