Introduction to Kazhdan's Property (T), II

We look at the following results mentioned in part I. and ex4.

<u>Thm</u> For a II_1 factor M, having property (T) implies fullness.

<u>Def</u> $\varphi \in \mathrm{Aut}(M)$ is <u>inner</u> if $\exists\, u \in \mathcal{U}(M)$, $\varphi(x) = uxu^*$, $\forall x \in M$.

$\mathrm{Inn}(M) = \{\varphi \in \mathrm{Aut}(M) \mid \varphi \text{ is inner}\}$

M is <u>full</u> if $\mathrm{Inn}(M)$ is closed in $\mathrm{Aut}(M)$.

(topology on $\mathrm{Aut}(M)$: $\varphi_i \to \varphi$ if $\lim_i \|\varphi_i(x) - \varphi(x)\|_2 = 0$ for all $x \in M$).

<u>Pf of Thm</u> Let (F, ε) be from property (T) of M.

Suppose $\varphi \in \mathrm{Aut}(M)$ satisfies $\max_{x \in F} \|\varphi(x) - x\|_2 < \varepsilon$.

Construct the M-M-bimodule $H(\varphi)$: the Hilbert space $L^2(M)$ with actions $x \cdot \xi \cdot y = \varphi(x)\xi y$, $\forall \xi \in L^2(M)$, $x, y \in M$.

Then for $\hat{1} \in H(\varphi)$, $\max_{x \in F} \|x \cdot \hat{1} - \hat{1} \cdot x\|_{H(\varphi)} = \|\varphi(x) - x\|_2 < \varepsilon$.

So there is a central vector $\eta \neq 0$ in $H(\varphi)$, and in $L^2(M)$, $\varphi(x)\eta = \eta x$, $\forall x \in M$. We may assume $\|\eta\| = 1$

Consider

$$\langle xy\eta, \eta\rangle = \langle x\eta\varphi^{-1}(y), \eta\rangle = \langle x\eta, \eta\varphi^{-1}(y^*)\rangle = \langle x\eta, y^*\eta\rangle = \langle yx\eta, \eta\rangle$$

By the uniqueness of trace on M, we have

$$< \cdot \eta, \eta > = \tau_M(\cdot), \quad \eta^* \eta = 1, \quad \eta \in U(M)$$

(polar decomposition $\eta = u|\eta|$, where $|\eta| = (\eta^*\eta)^{1/2} = 1$, so $\eta \in M$).

and $\varphi(x) = \eta x \eta^*$, $\forall x \in M$.

Therefore $\varphi \in \mathrm{Inn}(M)$. This shows that a nbhd of Id in $\mathrm{Aut}(M)$ is contained in $\mathrm{Inn}(M)$.

Then $\mathrm{Inn}(M)$ is an open (in particular closed) subgroup of $\mathrm{Aut}(M)$.

$\square$

It follows that the outer automorphism of M $\mathrm{out}(M) = \mathrm{Aut}(M)/\mathrm{Inn}(M)$ with quotient topology is discrete.

Connes showed this for M being a group factor in 1980. In the same paper Connes also showed that the fundamental group $\mathcal{F}(M) = \{t > 0 \mid M^t \simeq M\}$ is countable when M has property (T).

In ex 4 we use the following.

Thm Let M be a II_1 factor. Then M has property (T)
$\Rightarrow M$ has spectral gap: $\forall (x_n) \subset M$, if $\forall u \in \mathcal{U}(M)$,
$\lim_n \|u x_n - x_n u\|_2 = 0$, then $\lim_n \|x_n - \tau(x_n)\|_2 = 0$.

Pf for $M = L(G)$ where G is icc property (T):

We can assume $\tau(x_n) = 0$.

Let $\xi_n = x_n \delta_e \in \ell^2(G)$. Then $\langle \xi_n, \delta_e \rangle = 0$, for all n.

Consider the unitary representation of G on $\ell^2(G)$

$\pi : G \to \mathcal{U}(\ell^2(G))$, $\pi(g) = \lambda_g J \lambda_g J = \lambda_g \rho_g$

Restrict π to $\ell^2_0(G) = \{ \xi \in \ell^2(G) \mid \langle \xi, \delta_e \rangle = 0 \}$.

(For $\xi \in \ell^2_0(G)$, $\langle \pi(g)\xi, \delta_e \rangle = \langle \xi, \rho_g^* \lambda_g^* \delta_e \rangle = \langle \xi, \delta_e \rangle = 0$)

$\langle \pi(g)\xi_n, \xi_n \rangle = \langle \lambda_g x_n \delta_e, \rho_{g^{-1}} x_n \delta_e \rangle = \langle \lambda_g x_n \delta_e, x_n \lambda_g \delta_e \rangle$

Then
$$\begin{aligned}\|\pi(g)\xi_n - \xi_n\|^2 &= 2\|\xi_n\|^2 - 2\mathrm{Re}\langle \pi(g)\xi_n, \xi_n \rangle \\ &= 2\|x_n\|_2^2 - 2\mathrm{Re}\langle \lambda_g x_n, x_n \lambda_g \rangle \\ &= \|\lambda_g x_n - x_n \lambda_g\|_2^2 \longrightarrow 0\end{aligned}$$

Notice $\ell^2_0(G)$ does not contain any nonzero $\pi(G)$-invariant vector since G is icc. By property (T) it does not contain almost invariant vectors. We must have $\|\xi_n\| \to 0$, $\|x_n\|_2 \to 0$.

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