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Discrete dynamical systems spontaneously generated by a semigroup

1. Γ - discrete group.

(traditional) discrete dynamical system: $\Gamma \curvearrowright X$, $X = LCH$: Need X!

$\gamma \in \Gamma$, $x \in X$: $\gamma x \xleftarrow{(\gamma,x)} x$ one "element" of the homeo. γ

~~$r(\gamma,x)$ ... $s(\gamma,x)$~~

$G = \Gamma \times X$ is an étale groupoid (product topology)

$U \subseteq X$ open. $\gamma(U) \xleftarrow{\{\gamma\}\times U} U$ "partial homeo of X" : a "piece" of γ.

Sometimes this is all you have. — can still build a C^*-algebra like $C(X) \rtimes \Gamma$ (universal for representations of the "partial" dynamics) $C_c(\Gamma \times X)$

If X is totally disconnected (base of cpt-open sets), G is called ample

$X \supseteq U$ cpt-open : $\chi_{\{\gamma\}\times U}$ defines a partial isometry - total set in $C^*(G)$.

2. $\Lambda \subseteq \Gamma$ semigroup, $\Lambda \cap \Lambda^{-1} = \{e\}$

"less is more" : $\Lambda \rightsquigarrow \Lambda^* = $ cpt. T_2, $\Lambda \curvearrowright \Lambda^*$ by partial homeos.

$G(\Lambda)$ - corresponding ample groupoid.

How?? Special (important) case: LCM.

For $\alpha \in \Lambda$, $\alpha\Lambda$ = principal right ideal (all "extensions" of α)

$\alpha\Lambda \cap \beta\Lambda$ = common extensions of α, β. Write $\alpha \Cap \beta$ if $\alpha\Lambda \cap \beta\Lambda \neq \emptyset$

Λ is LCM if $\alpha \Cap \beta \Rightarrow \exists\, \gamma$ s.t. $\alpha\Lambda \cap \beta\Lambda = \gamma\Lambda$. (! min com. ext.)

Write $[\alpha] = \{\beta : \alpha \in \beta\Lambda\}$ - "prefixes" of α

<u>Def</u> (Nica): $x \subseteq \Lambda$ is <u>hereditary</u> if $\alpha \in x \Rightarrow [\alpha] \subseteq x$

<u>directed</u> if $\alpha, \beta \in x \Rightarrow \alpha\Lambda \cap \beta\Lambda \cap x \neq \emptyset$.

Thm..

Topology: For $\alpha \in \Lambda$, $\hat{Z}(\alpha) = \{x \in \Lambda^* : \alpha \in x\}$ "generalized cylinder set"

$\{\hat{Z}(\alpha) \setminus \hat{Z}(\beta)\}$ generate a tot. disc. cpt. T_2 topology on Λ^*

$\hat{Z}(\alpha)$ is cpt-open

Dynamics. For $\alpha \in \Lambda$, $x \in \Lambda^*$ put $\alpha x = \bigcup_{\beta \in x} [\alpha\beta]$.

$x \in X \mapsto \alpha x \in \hat{Z}(\alpha)$ is a homeo.

$G(\Lambda)$ — corresponding ample groupoid

Def. $\mathcal{T}C^*(\Lambda) := C^*(G(\Lambda))$ — "Toeplitz algebra" of Λ

$\partial\Lambda$ = closure of maximal elts of Λ^*

~~closed invariant subset~~

$\partial\Lambda \subseteq \Lambda^*$ closed invariant subset

$C^*(\Lambda) := C^*(G(\Lambda)|_{\partial\Lambda})$ — "CK algebra" of Λ

3. Examples (i) $\Lambda = \mathbb{N} \subseteq \mathbb{Z}$. Find Λ^*...

P.R.I.'s: $m + \Lambda = [m, \infty)$

$(m+\Lambda) \cap (n+\Lambda) = [\max\{m,n\}, \infty)$ — always nonempty (lattice order) LCM

$\Lambda^* = \{\overset{[n]}{\overset{\shortparallel}{[0,n]}}\} \cup \{\Lambda\}$

$\cong \mathbb{N} \cup \{\infty\}$

$\hat{Z}(n) = [n, \infty]$

$\hat{Z}(n) \setminus \hat{Z}(n+1) = \{n\}$ $\Big\}$ base of cpt-open sets

$+1$: $\overset{\in\Lambda^*}{[0,n]} \mapsto \overset{\in\Lambda^*}{[0,n+1]}$, $[0,\infty] \mapsto [0,\infty]$. I.e. $+1$ on $\tilde{\mathbb{N}}$.

$\partial\Lambda = \{\infty\}$ — $\mathbb{N}$ acts trivially on $\partial\Lambda$.

$C^*(\mathbb{N}) \cong C^*(G(\mathbb{N})|_{\{\infty\}}) \cong C^*(\mathbb{Z}) \cong C(\mathbb{T})$

$\{\infty\}^c \cong \mathbb{N}$ open, invariant, transitive, principal

$C^*(G(\Lambda)|_{\mathbb{N}}) \cong \mathcal{K}(\ell^2(\mathbb{N}))$

$(+1 : n \mapsto n+1) = e_{n+1,n}$ matrix unit.

$0 \to \mathcal{K} \to \mathcal{T}C^*(\mathbb{N}) \to C(\mathbb{T}) \to 0$

generated by $+1$ on $\mathbb{N} \cup \{\infty\}$ – unilateral shift – usual Toeplitz algebra.

(ii) $P = \mathbb{Z}[\frac{1}{2}] = \{\frac{m}{2^k} : m \in \mathbb{Z},\ k \in \mathbb{N}\}$

$\Lambda = \mathbb{Z}[\frac{1}{2}]_+ = \{\lambda \in P : \lambda \geq 0\}$.

Again totally ordered hence lattice ordered, LCM

Λ^*: For $\lambda \in \Lambda$, $[\lambda] = [0, \lambda] =: \lambda^+$

For $t \in \mathbb{R}_{++}$, $[0, t) =: t$

$\Lambda =: \infty$

$[0, \infty]$ but points of $\Lambda \setminus \{0\}$ are doubled (e.g. $\frac{1}{2}\ (\frac{1}{2})^+$)

For $\lambda \in \Lambda$, $[0, \lambda) \subsetneq [0, \lambda]$, i.e. $\lambda < \lambda^+$

$\lambda_1 < \lambda_2$: $[0, \lambda_1] \subseteq [0, \lambda_2)$, i.e. $\lambda_1^+ < \lambda_2$

etc.

$\mathcal{Z}(\lambda) = \{\mu^+ : \mu \geq \lambda\} \sqcup \{t : t > \lambda\} \cup \{\infty\} =: [\lambda^+, \infty]$

$\lambda_1 < \lambda_2$: $\mathcal{Z}(\lambda_1) \setminus \mathcal{Z}(\lambda_2) = \{\mu^+ : \lambda_1 \leq \mu < \lambda_2\} \sqcup \{t : \lambda_1 < t \leq \lambda_2\} =: [\lambda_1^+, \lambda_2]$

$\partial\Lambda = \{\infty\}$. $C^*(\Lambda) = C^*(G(\Lambda)|_{\{\infty\}}) \cong C^*(\mathbb{Z}[\frac{1}{2}]) \cong C(X_{2^\infty})$

2-adic solenoid

$I = C^*(G(\Lambda)|_{\{\infty\}^c})$: $0 \to I \to \mathcal{T}C^*(\Lambda) \to C(X_{2^\infty}) \to 0$

$+1 : [n^+, n+1) \to [(n+1)^+, n+2)$: $e^{(0)}_{n+1,n}$ matrix unit $\mathcal{K}(\ell^2(\mathbb{N}))$

$+\frac{1}{2} : [(\frac{n}{2})^+, \frac{n+1}{2}) \to [(\frac{n+1}{2})^+, \frac{n}{2}+1)$: $e^{(1)}_{\frac{n+1}{2}, \frac{n}{2}}$ $\mathcal{K}(\ell^2(\frac{1}{2}\mathbb{N}))$

etc.

$\mathcal{K}(\ell^2(\mathbb{N})) \subseteq \mathcal{K}(\ell^2(\tfrac{1}{2}\mathbb{N})) \subseteq \cdots \subseteq \mathcal{I} = \overline{\bigcup_k \mathcal{K}(\ell^2(\tfrac{1}{2^k}\mathbb{N}))}$

multiplicity two embeddings ($e^{(0)}_{00} = e^{(1)}_{00} + e^{(1)}_{\frac{1}{2}\frac{1}{2}}$)

$\mathcal{I} \cong \mathcal{K} \otimes UHF(2^\infty)$.

n^th stage: $0 \to \mathcal{K}(\ell^2(\tfrac{1}{2^n}\mathbb{N})) \to \mathcal{T}_n \to C^*(\tfrac{1}{2^n}\mathbb{Z}) \to 0$

$\mathcal{T}C^*(\Lambda) = \overline{\bigcup_n \mathcal{T}_n}$.

(iii) $\mathbb{Z} \overset{\cdot 2}{\curvearrowright} \mathbb{Z}[\tfrac{1}{2}]$. $\Gamma = \mathbb{Z}[\tfrac{1}{2}] \rtimes_2 \mathbb{Z}$

$\mathbb{Z}[\tfrac{1}{2}] = \langle \ldots, c_{-1}, c_0, c_1, c_2, \ldots \mid c_n^2 = c_{n+1} \rangle$ $\quad (c_n \equiv 2^n)$

$\Gamma = \langle \{c_n : n \in \mathbb{Z}\} \cup \{a\} : c_n^2 = c_{n+1},\ a c_n a^{-1} = c_n^2 \rangle$

$= \langle c_0, a \mid a c_0 a^{-1} = c_0^2 \rangle$

$= \langle a, b \mid ab = b^2 a \rangle$ Baumslag-Solitar group (1,2)

(Choose $\Lambda = \langle a, b \mid ab = b^2 a \rangle_+$ (use only nonnegative powers of generators).

Normal form in Λ: $b^{i_0} a b^{i_1} a \cdots b^{i_{m-1}} a b^p$, $i_0, \ldots, i_{m-1} \in \{0,1\}$, $p \in \mathbb{N}$.

Fact: $b^{i_0} a \cdots b^{i_{m-1}} a b^p \Cap b^{j_0} a \cdots b^{j_{n-1}} a b^q$

iff $(i_0, \ldots, i_{m-1})$ and $(j_0, \ldots, j_{n-1})$ are comparable (one extends the other)

Λ is LCM, not lattice ordered.

What is Λ^*? For $i \in \prod_0^\infty \{0,1\}$, $x(i) = \bigcup_{m,p \geq 0} [b^{i_0} a \cdots b^{i_{m-1}} a b^p]$.

$\partial\Lambda = \{x(i)\} \cong \prod_0^\infty \{0,1\}$.

$\Lambda \curvearrowright \partial\Lambda$: b acts as the "odometer": $(1,0,0,\ldots) + (i_0, i_1, \ldots)$, carry to ~~+1 and carry to right~~ right

homeo!

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(5)

$C(\partial\Lambda) \rtimes_\sigma \mathbb{Z} = BD(2^\infty)$ Bunce-Deddens algebra
(1-dim'l version of $UHF(2^\infty)$)

$a \cdot (i_0, i_1, \ldots) = (0, i_0, i_1, \ldots)$ right shift

$C^*(\Lambda) = C^*(G(\Lambda)|_{\partial\Lambda}) \cong (C(\partial\Lambda) \rtimes_\sigma \mathbb{Z}) \rtimes_\alpha \mathbb{N}$ - UCT Kirchberg
$K_i \cong \mathbb{Z}$

$(\partial\Lambda)^c$ open invariant : $I = C^*(G(\Lambda)|_{(\partial\Lambda)^c})$
$0 \to I \to \mathcal{T}C^*(\Lambda) \to C^*(\Lambda) \to 0$
$\uparrow$ identify

$\Lambda^* \setminus \partial\Lambda$: For $\lambda \in \Lambda$, $[\lambda] = \mathcal{Z}(\lambda) \setminus (\mathcal{Z}(\lambda a) \cup \mathcal{Z}(\lambda b))$ - open point.

$X_0 = \{[\lambda] : \lambda \in \Lambda\}$ discrete open invariant, transitive, principal

$C^*(G(\Lambda)|_{X_0}) \cong \mathcal{K}(\ell^2(\Lambda))$. $0 \to \mathcal{K}(\ell^2(\Lambda)) \to I \to C^*(G)|_{(X_0 \cup \partial\Lambda)^c}$
$\to 0$

$\Lambda^* \setminus (X_0 \cup \partial\Lambda) = X_1 \sqcup X_2$: invariant relatively open in X_0^c.

X_1: $B = \langle b \rangle = \{b^n : n \geq 0\}$.
For $\alpha \in \Lambda$, $\alpha B = \bigcup_{n \geq 0} [\alpha b^n] \in \Lambda^*$

$X_1 \cong \Lambda/_B := \{\alpha B : \alpha \in \Lambda\}$.
$G(\Lambda)|_{X_1}$ is transitive, isotropy $\cong \mathbb{Z}$
$C^*(G(\Lambda)|_{X_1}) \cong \mathcal{K}(\ell^2(\Lambda/_B)) \otimes C^*(\mathbb{Z})$

X_2: $A = \langle a \rangle = \{a^n : n \in \mathbb{N}\}$

$\alpha A = \bigcup_{n \geq 0} [\alpha a^n] \subset \Lambda^*$

$X_2 = \Lambda/A = \{\alpha A : \alpha \in \Lambda\}$

$G(\Lambda)|_{X_2}$ transitive, isotropy $\mathbb{Z}$

$C^*(G(\Lambda)|_{X_2}) \cong \mathcal{K}(\ell^2(\Lambda/A)) \otimes C^*(\mathbb{Z})$

So $0 \to \mathcal{K}(\ell^2(\Lambda)) \to I \to \left(\mathcal{K}(\ell^2(\Lambda/A)) \otimes C(\mathbb{T})\right) \oplus \left(\mathcal{K}(\ell^2(\Lambda/B)) \otimes C(\mathbb{T})\right) \to 0$

$0 \to I \to \mathcal{T}C^*(\Lambda) \to C^*(\Lambda) \to 0$

References ~~X~~ Li in Oberwolfach Seminars 47

JS: categories of paths JAMS 2014 (in §8)