GOALS

Interpolated Free Group Factors are Group Factors

(joint w. S. Popa)

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$L(\mathbb{F}_n) = \lambda(\mathbb{F}_n)'' \subseteq B(\ell^2(\mathbb{F}_n))$

$\lambda : \mathbb{F}_n \to \mathcal{U}(\ell^2(\mathbb{F}_n))$

$\mathbb{F}_n$ ICC (inf. conj. classes)

$\hookrightarrow$ $L(\mathbb{F}_n)$ factors

$\langle \delta_e, \cdot\, \delta_e \rangle$ trace on $L(\mathbb{F}_n)$

$\hookrightarrow$ II_1 factors

M II_1 factor $\tau : M \to \mathbb{C}$ trace

τ is unique linear functional $\varphi : M \to \mathbb{C}$

s.t. $\varphi(xy) = \varphi(yx)$ $\forall x, y \in M$

$\varphi(1) = 1$ and $\varphi(x^* x) \geq 0$

$\forall x \in M$

$$M / [M,M] \overset{\tau}{\cong} \mathbb{C}$$

$$\ell^2 \cong \ell^2(\mathbb{N})$$

$\mathcal{M} := M \otimes B(\ell^2)$ II_∞ factor $\tau_M \otimes Tr_{B(\ell^2)}$

$p, q \in \mathcal{M}$ projection $\begin{pmatrix} \exists u \\ upu^* = q \end{pmatrix} \Leftrightarrow \tau_M \otimes Tr_{B(\ell^2)}(p) = \tau_M \otimes Tr_{B(\ell^2)}(q)$

Choose $t \in (0,+\infty)$, choose $p \in \mathcal{M} = M \otimes B(\ell^2)$

s.t. $\tau_M \otimes Tr_{B(\ell^2)}(p) = t$

let $M^t = p \mathcal{M} p$ ← isom class depends only on t

II_1 fact with trace $\frac{1}{t} \tau_M \otimes Tr_{B(\ell^2)}(\cdot)$

$(M^t)^s = M^{ts}$

$M \longrightarrow M^t$ amplification

$\underline{Ex}$: if $t = n \in \mathbb{N} \Rightarrow M^t \cong M_{n \times n}(M)$

$t < 1$ $\quad M^t = qMq \quad q \in M,\ \tau(q) = t.$

Fundamental group of M

$$\mathcal{F}(M) = \{t : M^t \cong M\} \subseteq \boxed{\mathbb{R}_+}$$

$$\left[\ \mathcal{F}(M) = \{t \in \mathbb{R}_+ : \exists\, \alpha \in \mathrm{Aut}(M \otimes B(\ell^2)) \text{ s.t. } \tau_M \otimes \mathrm{Tr}_{B(\ell^2)} \circ \alpha = t \cdot \tau_M \otimes \mathrm{Tr}_{B(\ell^2)}\}\ \right]$$

Q: what can $\mathcal{F}(M)$ be? Open

Popa – Vaes: can construct M with $\mathcal{F}(M)$ any a big list of groups.

1st result (Murray - von Neumann):

$\mathcal{F}(R) = \mathbb{R}$

R - hyperfinite II_1 factor

$R = L(S_\infty)$

$S_\infty = \bigcup_N S_N$

40s

90s

Result (Voiculescu): $\mathcal{F}(L(\mathbb{F}_\infty)) \supseteq \mathbb{Q}_+$

(Radulescu) $\mathcal{F}(L(\mathbb{F}_\infty)) = \mathbb{R}_+$

Open: $L(\mathbb{F}_n)$ $n < \infty$?

Voiculescu

Th (Dykema- Radulescu)

Free group factor isomorphism question

Either $\mathcal{F}(L(\mathbb{F}_n)) = \{1\}$

(and then $L(\mathbb{F}_n) \not\cong L(\mathbb{F}_m)$ if $n \neq m$, $n, m \in \mathbb{N} \cup \{\infty\}$)

Or $\mathcal{F}(L(\mathbb{F}_n)) = \mathbb{R}_+$

(and then $L(\mathbb{F}_n) \cong L(\mathbb{F}_\infty)$ $\forall n \in \mathbb{N}$).

Interpolated free group factors.

Th (Voiculescu, Dykema, Radulescu)

$$L(\mathbb{F}_n)^s \cong L(\mathbb{F}_m) \quad \text{if}$$

$$\frac{1}{s^2}(n-1) = (m-1)$$

Defn $\boxed{L(\mathbb{F}_t)} = L(\mathbb{F}_n)^s$ where s is such

that $(t-1) = \frac{1}{s^2}(n-1)$

Free products:

(M_1, φ_1) (M_2, φ_2) v. N. alg.

$\hookrightarrow \exists$ $(\underbrace{M_1 * M_2}_{M}, \varphi)$ s.t.

$M_1, M_2 \leq M$ freely independent.

$$(L(G_1), \tau_{G_1}) * (L(G_2), \tau_{G_2}) = L(G_1 * G_2, \tau_{G_1 * G_2})$$

Nice theorems:

wrt traces

- $L(\mathbb{F}_s) * L(\mathbb{F}_t) \cong L(\mathbb{F}_{s+t})$

- if M_i are amenable (e.g. f.dim. abelian hyperfinite) / int. free group factors

then $*(M_i) \cong L(\mathbb{F}_s)$

+ formula for s.

$$L(S_\infty * \mathbb{F}_2 * \mathbb{Z}/3\mathbb{Z}) \cong L(\mathbb{F}_{3\,2/3})$$

?

[Dykema et al]

- $\mathcal{M}$ $\quad \Gamma = \Gamma_1 *_{A_1} \Gamma_2 *_{A_2} \cdots \Gamma_n \qquad |A_j| < \infty$

Γ_j is amenable or a free group

if a factor

$\Rightarrow \quad L(\Gamma) \cong L(\mathbb{F}_t)$

$t = 1 + \beta_1^{(2)}(\Gamma)$

What Γ have the property that $L(\Gamma) \cong L(\mathbb{F}_t)$?

Which groups produce interpolated free group factors?

$$L(\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/7\mathbb{Z} * \mathbb{Z} * \mathbb{F}_2)$$

$$\cong L(\mathbb{F}_t) \qquad t = \tfrac{1}{2} + \tfrac{6}{7} + 1 + 2$$

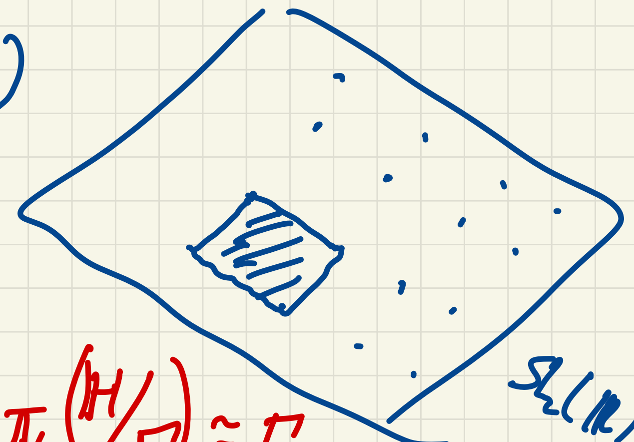
Conj (Voiculescu and de la Harpe)

$\Gamma \le PSL_2(\mathbb{R})$ lattice $\quad \det\begin{bmatrix} * & * \\ * & * \end{bmatrix}=1 \Big/ \text{center}$

then $L(\Gamma) \cong L(\mathbb{F}_t) \quad t = f(\mathrm{covol}(\Gamma))$.

Ex: $\Gamma = PSl_2(\mathbb{Z}) \le PSL_2(\mathbb{R})$

$L(\Gamma) \cong L(\mathbb{F}_{7/6})$



Very open for Γ surface group $\pi_1(\mathbb{H}/\Gamma) \cong \Gamma$

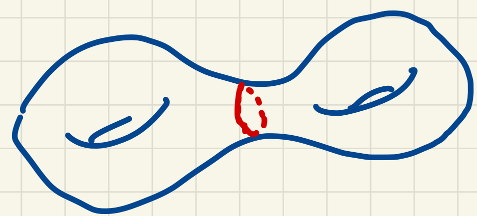
$\Gamma \le PSL_2(\mathbb{R})$ co-compact.

Can find examples

$$\Gamma_1 \underset{\Gamma_2}{*} \Gamma_3$$

not free, int group finite.

with Γ_2 amenable



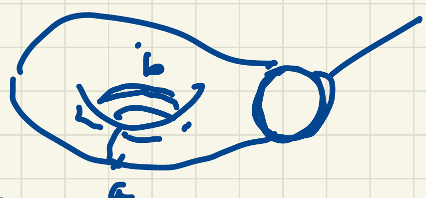
$$\Gamma = \pi_1(\text{surface})$$

$$\mathbb{F}_2 \underset{\mathbb{Z}}{*} \mathbb{F}_2$$

$$\Gamma = \Gamma_1 * \Gamma_2$$

$\Gamma_1 = \langle a, b \rangle$

$\mathbb{Z} = \langle a^{-1}bab^{-1} \rangle$



Q (w. S. Popa): Can we find G s.t. $\forall t$

$$L(G) \cong L(\mathbb{F}_t)?$$

Conj: such G cannot be fin. gen if $t \notin \mathbb{Q}_+$.

Yes if $t = n$ $\quad G = \mathbb{F}_n$

$t = 7/6$ $\quad G = PSL_2(\mathbb{Z})$

$\vdots$

G. Hjorth
ME
side

Idea: construct G_t s.t. $L(G_t) \cong L(\mathbb{F}_t)$

G_t inductive limit.

Fix finite groups $H_1 \subseteq H_2 \subseteq \cdots$ $n_1, n_2, \ldots$ integers (may be 0)

$\bigcup H_j$ ICC (e.g. $H_j = S_j$

$W^*(\bigcup H_j) \cong R$ operator-valued semicircular systems.

$G_1 = H_1 * \mathbb{F}_{n_1}$ $G_2 = G_1 *_{H_1} (H_1 \times \mathbb{F}_{n_2})$

$G_k = G_{k-1} *_{H_{k-1}} (H_{k-1} \times \mathbb{F}_{n_k})$

Th. $G = \bigcup G_n$ $L(G) \cong L(\mathbb{F}_t)$

$$t = 1 + \sum \frac{n_j}{|H_{j-1}|} = 1 + \sum \frac{n_j}{(j-1)!}$$

$H_j = S_j$ can choose n_j's s.t.

$t =$ any number in $(1, +\infty)$

been $\frac{1}{j!} \to 0$.

$$L(\mathbb{F}_n) \le L(\mathbb{F}_2)$$

$\Gamma \le \mathbb{F}_n$ $[\mathbb{F}_n : \Gamma] = k$

$\Gamma \cong \mathbb{F}_{k(n-1)+1}$

Schreier formula

Open: $M \le L(\mathbb{F}_n)$

subfactor is $M \cong L(\mathbb{F}_t)$

$\lambda = [L(\mathbb{F}_n) : M]$ $t - 1 = \lambda(n-1)$

OE of actions.

$\Gamma_1, \Gamma_2 \curvearrowright (X, \mu)$

$\Gamma_1 \overset{OE}{\cong} \Gamma_2$ if actions have same orbits.

R_{Γ_1} $x \underset{R_{\Gamma_1}}{\sim} y \iff x = \alpha_g(y)$ for some $g \in \Gamma_1$

Gaboriau

$\mathbb{F}_n \curvearrowright (X, \mu) \rightsquigarrow R$

$C(R) = n$

$\alpha \in \mathrm{Aut}(L(\mathbb{F}_n))$ of period 2 $\quad \alpha \circ \alpha = \mathrm{id}$

(resp. outer)

$$M = \{ x \in L(\mathbb{F}_n) : \alpha(x) = x \}$$

is $M \cong L(\mathbb{F}_{1+2(n-1)})$?