Graph algebras, groupoids, and subalgebras

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Groundwork for Operator Algebras Lecture Series
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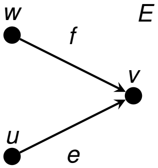
A **directed graph** is a four-tuple $E = (E^0, E^1, r, s)$ consisting of

- a countable set E^0 – the vertices of E,
- a countable set E^1 – the edges of E, and
- maps $r, s : E^1 \to E^0$ called the range and source maps.

Notation: Denote by E^n the set of paths of length $n \in \mathbb{N}^+$, and $E^* = \cup_{n=0}^{\infty} E^n$.

Example



$$E^0 = \{u, v, w\}$$
$$E^1 = \{e, f\}$$

$$s(e) = u \quad r(e) = v$$
$$s(f) = w \quad r(f) = v$$

Let $E = (E^0, E^1, r, s)$ be a directed graph and H a Hilbert space.

A **Cuntz-Krieger E-system** on H is a collection of

- orthogonal projections $\{P_v \mid v \in E^0\}$ onto subspaces of H, and
- partial isometries $\{S_e \mid e \in E^1\}$ on H, s.t.

(i) The projections $P_v,\ v \in E^0$, are mutually orthogonal,

(ii) for $e \in E^1$, $S_e^* S_e = P_{s(e)}$ and $S_e S_e^* \leq P_{r(e)}$, and

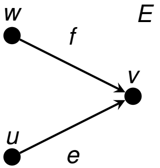
(iii) for $v \in E^0$, $\displaystyle\sum_{r(e)=v} S_e S_e^* = P_v$.

Note: we assume $\forall v \in E^0,\ 0 < |r^{-1}(\{v\})| < \infty$.

Conventions: For a path $\lambda = e_1 e_2 \cdots e_n$ in E^n, denote $S_\lambda = S_{e_1} S_{e_2} \cdots S_{e_n}$. Consider vertices $v \in E^0$ to be paths of length zero, and denote $S_v = P_v$.

$$C^*(S_\lambda) = \text{the } C^*\text{-algebra in } \mathcal{B}(H) \text{ generated by these operators}$$
$$= \{S_\alpha S_\beta^* \mid \alpha, \beta \text{ paths with } s(\alpha) = s(\beta)\} \text{ (exercise)}$$

Example



A CK-system for E requires

- projections P_w, P_v, P_u,
- partial isometries

$$S_f : P_w H \to P_v H$$
$$S_e : P_u H \to P_v H,$$

- $P_v H = S_e(P_w H) \oplus S_f(P_u H)$.

We can find such a system on $H = \mathbb{C}^4$:

$$P_w = \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \quad P_u = \begin{pmatrix} 0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \quad P_v = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}$$

$$S_f = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\1&0&0&0\\0&0&0&0 \end{pmatrix} \quad S_e = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&0 \end{pmatrix}$$

Fact: Given a directed graph E, there exists a *universal* Cuntz-Krieger E-system (p_v, s_e), meaning that $C^*(E) := C^*(p_v, s_e)$ satisfies the following.

For any E-system (P_v, S_e) there is a unique $*$-homomorphism

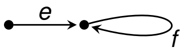
$$\begin{aligned} \pi : C^*(E) &\to C^*(P_v, S_e) \\ p_v &\mapsto P_v \\ s_e &\mapsto S_e. \end{aligned}$$

Two immediate questions:

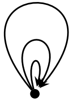
1. Under what conditions can we be sure that $C^*(E) \cong C^*(P_v, S_e)$? (uniqueness theorems)
2. Under what conditions on graphs E and F can we be sure that $C^*(E) \cong C^*(F)$? (classification by moves)

Examples

•Toeplitz algebra



• Cuntz Algebras ('77)



(n loops: $\mathcal{O}_n$)

• Cuntz-Krieger algebras ('80)

0-1 adjacency matrix A
$\rightsquigarrow$ Cuntz-Krieger algebra $\mathcal{O}_A$

• Finite-dimensional C^*-algebras.

• The compact operators on a separable Hilbert space.

• Approximately finite (AF) algebras – Morita equivalent to graph algebras.

(Recall that Morita equivalence is an equivalence relation on C^*-algebras that captures when they have the same representation theory.)

- Kumjian-Pask-Raeburn: $C^*(E)$ is AF iff E has no cycles.

Ideals:

- Gauge-invariant ideals correspond to saturated hereditary subsets of vertices. Bates-Pask-Raeburn-Szymański (2000): Quotients by these ideals produce C^*-algebras of quotient graphs.
- The ideal structure of $C^*(E)$ can be completely described from the graph, for arbitrary E. (Hong-Szymański (2004)).
- Brown-Fuller-Pitts-R (2020): Quotients by regular ideals preserve Condition (L) (see next slide) in the graph.

Question 2: When is $C^*(E) \cong C^*(F)$?

Eilers-Restorff-Ruiz-Sørensen (2016): a complete list of moves on graphs classifying graph algebras up to Morita equivalence.

Eilers-Ruiz (2019): moves on graphs preserving other notions of equivalence, including isomorphism.

Question 1: If $\{S_\lambda,\ \lambda \in E^*\}$ is a Cuntz-Krieger E-system, when is $C^*(S_\lambda)$ isomorphic to $C^*(E)$?

- The system is *nondegenerate* if every range projection $S_\lambda S_\lambda^* \neq 0$.
- The graph satisfies Condition (L) if every cycle has an entry. That is, for every path $e_1 e_2 \ldots e_n$ with $r(e_1) = s(e_n)$ there exists an i and an edge $e \neq e_i$ such that $r(e) = r(e_i)$.

Cuntz-Krieger Uniqueness Theorem (Kumjian-Pask-Raeburn-Fowler, '90s) When E satisfies condition Condition (L) then for any nondegenerate Cuntz-Krieger E-system $\{S_\lambda\}$, $C^*(S_\lambda) \cong C^*(E)$.

In other words, when E satisfies Condition (L), then the following are equivalent:

(i) The canonical $*$-homomorphism $\pi : C^*(E) \to C^*(S_\lambda)$ is injective.

(ii) π is injective on the diagonal subalgebra $\mathcal{D} := C^*(\{s_\alpha s_\alpha^* \mid \alpha \in E^*\})$ of range projections of the paths.

Nagy-Reznikoff (2012): Let $\{S_\lambda, \lambda \in E^*\}$ be a Cuntz-Krieger E-system. Then the following are equivalent.

(i) The canonical $*$-homomorphism $\pi : C^*(E) \to C^*(S_\lambda)$ is injective.

(ii) π is injective on the *cycline subalgebra* $\mathcal{M} := C^*(\{s_\alpha s_\beta^* \mid \alpha, \beta \in E^*,\ \alpha \sim \beta\})$,

where $\alpha \sim \beta$ if $\alpha = \beta$ or there is a cycle without entry λ s.t. $\beta = \alpha\lambda$ or $\alpha = \beta\lambda$.

Features of the cycline subalgebra:

- $\mathcal{M}$ is a maximal abelian self-adjoint subalgebra (*masa*)
- there is a faithful conditional expectation $\mathbb{E} : C^*(E) \to \mathcal{M}$
- $\{x \in C^*(E) \mid x\mathcal{M}x^* \cup x^*\mathcal{M}x \subseteq \mathcal{M}\}$ is dense in $C^*(E)$.
- The set of pure states on $\mathcal{M}$ that extend uniquely to pure states on $C^*(E)$ is weak-$*$ dense in the state space.

(Recall: a *state* is a positive linear functional of norm 1. A pure state is an extreme point in the state space.)

The first three items mean that $\mathcal{M}$ is a **Cartan subalgebra** of $C^*(E)$. This brings us to the topic of groupoids.

Recall that a groupoid is a nonempty small category $\mathcal{G}$ with inverses. That is:

- a set-sized collection of morphisms and objects
- source and range (target) maps denoted s and r from $\mathcal{G}$ to the set of objects, denoted $\mathcal{G}^{(0)}$,
- an associative composition, with gh defined whenever $r(h) = s(g)$, and s.t. $r(gh) = r(g)$, $s(gh) = s(h)$.
- an inverse operation assigning to any $g \in \mathcal{G}$ an element $g^{-1} \in \mathcal{G}$ defined by $gg^{-1} = r(g)$ and $g^{-1}g = s(g)$.

The **path groupoid** of a directed graph E:

Let $E^\infty = \{x_1x_2\dots \mid \forall n \in \mathbb{N}^+ \; x_1x_2\dots x_n \in E^*\}$.

$$\mathcal{G}_E = \{(\alpha y,\ \ell(\alpha) - \ell(\beta),\ \beta y) \mid y \in E^\infty,\ \alpha, \beta \in E^*\},$$

$$s(x, d, z) = (z, 0, z), \quad r(x, d, z) = (x, 0, x)$$

$$(x, m, y)^{-1} = (y, -m, x) \quad (x, m, y)(y, n, z) = (x, m + n, z)$$

The cylinder sets $Z(\alpha, \beta) = \{(\alpha y, \ell(\alpha) - \ell(\beta), \beta y) \mid y \in E^\infty\}$ form a basis for a locally compact Hausdorff étale topology.

Recall: the C^*-algebra of a groupoid is a completion of the space of compactly supported continuous functions. (Robin Deeley's slides 38–40)

For a directed graph E, $C^*(E) \cong C^*(\mathcal{G}_E)$, via the map $s_\alpha s_\beta^* \mapsto \chi_{Z(\alpha,\beta)}$.

What is the image of the cycline subalgebra $\mathcal{M} = C^*(s_\alpha s_\beta^* \mid \alpha \sim \beta\}$?

Recall: $\alpha \sim \beta$ iff $\alpha = \beta$ or $\beta = \alpha\lambda$ (or vv) where λ is a cycle without entry.

Observation: $\alpha \sim \beta$ iff $\forall y \in E^\infty$, $\alpha y = \beta y$. Therefore,

- $\alpha \sim \beta$ iff for all $g = (\alpha y, \ell(\alpha) - \ell(\beta), \beta y) \in Z(\alpha, \beta)$, we have $r(g) = \alpha y = \beta y = s(g)$.
- $\alpha \sim \beta$ iff $Z(\alpha, \beta) \subseteq \mathrm{Iso}(\mathcal{G}_E) := \{g \in \mathcal{G}_E \mid r(g) = s(g)\}$.
- Thus the image of $\mathcal{M} := C^*(\{s_\alpha s_\beta^* \mid \alpha \sim \beta\})$ in $C^*(\mathcal{G}_E)$ under the isomorphism above is $C^*(\mathrm{Iso}(\mathcal{G}_E)^\circ)$.

Theorem (Brown-Nagy-R-Sims-Williams, 2014)
For a locally compact Hausdorff étale groupoid G, a $*$-homomorphism $\phi : C^*(G) \to B(H)$ is injective iff it is injective on $C^*(\mathrm{Iso}(G)^\circ)$.

A typical graph algebra is not abelian, as $s_\alpha s_\beta \neq 0$ requires $r(\beta) = s(\alpha)$.

A fruitful method of studying nonabelian operator algebras is to examine nice abelian subalgebras, such as Cartan subalgebras. We saw that the cycline subalgebra of a graph algebra is Cartan.

A brief history of Cartan subalgebras:

1971 Vershik: notion of Cartan sub-von Neumann algebra.

1977 Feldman-Moore: Cartan von Neumann pairs arise from measured countable equivalence relations.

1980 Renault's definition of Cartan C^*-subalgebras. Correspond to topologically principal étale groupoid with a twist.

1986 Kumjian: notion of C^*-diagonal, corresponding to subalgebra pairs arising from twisted equivalence relations.

A *twist* is an extension of groupoids: an exact sequence

$$\mathbb{T} \times G^{(0)} \xrightarrow{\iota} \Sigma \xrightarrow{q} G \quad \text{such that}$$

- q and ι are continuous groupoid homomorphisms (homeomorphisms of the unit spaces), ι is injective.
- $q^{-1}(G^{(0)}) = \iota(\mathbb{T} \times G^{(0)})$ • $\Sigma/\mathbb{T} \cong G$.

The C^*-algebra $C_r^*(\Sigma; G)$ of the twist is a completion of

$$C_c(\Sigma; G) := \{f \in C_c(\Sigma) \,|\, \forall z \in \mathbb{T} \ \ \forall \gamma \in \Sigma \ \ f(z \cdot \gamma) = \overline{z} f(\gamma)\}.$$

Renault ('08): Cartan subalgebras $\mathcal{B} \subseteq \mathcal{A}$ correspond to étale, 2^{nd} countable, locally compact Hausdorff, topologically principal twisted groupoids: $(\mathcal{A}, \mathcal{B}) \cong (C_r^*(\mathcal{G}; \Sigma), C_0(\mathcal{G}^{(0)}))$.

Recap: From a directed graph E we obtain a C^*-algebra, $C^*(E)$.

- $C^*(E)$ has a groupoid representation, $C^*(\mathcal{G}_E)$.
- $C^*(E)$ has a useful Cartan subalgebra, the cycline subalgebra $\mathcal{M}$.
- $\mathcal{M}$ is the C^*-algebra of the subgroupoid $\mathrm{Iso}(\mathcal{G}_E)^\circ$.

However:

- The groupoid from Renault's theorem is not typically the path groupoid!
- The cycline subalgebra of a topological groupoid algebra is not always Cartan.
- Topological groupoids can give rise to other Cartan subalgebras.

Cartan subalgebras that arise from other subgroupoids:

Theorem (Duwenig-Gillaspy-Norton-R-Wright, 2019)
Let G be a second countable, locally compact Hausdorff, étale groupoid, and $c : G^{(2)} \to \mathbb{T}$ a 2-cocycle. Assume S is maximal among abelian subgroupoids of $\mathrm{Iso}(G)$ on which c is symmetric. If S is clopen, normal, and immediately centralizing, then $C^*_r(S, c)$ is a Cartan subalgebra of $C^*_r(G, c)$.

Twisted groupoid representations of C^*-algebras arising from “relatively Cartan” subalgebras:

Theorem (Brown-Fuller-Pitts-R, 2018): Certain twists correspond to subalgebras $D \subseteq B \subseteq A$ where (B, D) is a Cartan inclusion and $D \subseteq A$ is a “homogenous regular inclusion”.

Thank you!

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