# Operator Algebra Techniques in Quantum Information Theory

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# The Setting

- $\mathcal{H}$ is a Hilbert space
- $\mathfrak{B}(\mathcal{H})$ is the algebra of all bounded operators on $\mathcal{H}$
- $\mathcal{T}(\mathcal{H})$ is the Banach space of all trace-class operators: all operators in $\mathfrak{B}(\mathcal{H})$ that have a finite trace
- The convex subset $\mathcal{S}(\mathcal{H}) \subset \mathcal{T}(\mathcal{H})$ of all positive ($\langle \rho\xi, \xi \rangle \geq 0 \ \forall \xi \in \mathcal{H}$), trace-one trace-class operators $\rho$ (called *states* or density operators)

# Positive, Unital Maps

**Definition**

Let $\phi : \mathfrak{B}(\mathcal{H}_1) \to \mathfrak{B}(\mathcal{H}_2)$ be a linear transformation. Then $\phi$ is

1. *positive* if it sends positive elements to positive elements: $\phi(A) \geq 0 \ \forall A \geq 0$;
2. *unital*, if $\phi(I_{\mathfrak{B}(\mathcal{H}_1)}) = I_{\mathfrak{B}(\mathcal{H}_2)}$ (i.e., $\phi$ maps the identity to the identity)

# Completely Positive Maps

## Definition

A linear transformation $\phi : \mathfrak{B}(\mathcal{H}_1) \to \mathfrak{B}(\mathcal{H}_2)$ *induces* a linear transformation $\phi^{(n)} : M_n(\mathfrak{B}(\mathcal{H}_1)) \to M_n(\mathfrak{B}(\mathcal{H}_2))$ defined by

$$\phi^{(n)}\left([r_{ij}]_{i,j=1}^n\right) = [\phi(r_{ij})]_{i,j=1}^n ,$$

i.e. $\phi^{(n)} : M_n(\mathbb{C}) \otimes \mathfrak{B}(\mathcal{H}_1) \to M_n(\mathbb{C}) \otimes \mathfrak{B}(\mathcal{H}_2)$, $\phi^{(n)} = \mathrm{id}_n \otimes \phi$.
We say that $\phi$ is

1. *n-positive* if

   $$\phi^{(n)}(X) \text{ is positive in } M_n(\mathfrak{B}(\mathcal{H}_2)) \text{ for every positive } X \in M_n(\mathfrak{B}(\mathcal{H}_1));$$

2. *positive*, if $\phi$ is $n$-positive for $n = 1$;
3. *completely positive*, if $\phi$ is $n$-positive for every $n \in \mathbb{N}$;
4. *UCP* if $\phi$ is unital and completely positive.

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# Quantum Channels

## Definition

A *quantum channel* is a completely positive, trace-preserving (CPTP) linear map $\Phi : \mathcal{T}(\mathcal{H}_1) \to \mathcal{T}(\mathcal{H}_2)$.

Notes:

1. trace-preserving means $\mathrm{Tr}(\phi(X)) = \mathrm{Tr}(X)\ \forall X \in \mathcal{T}(\mathcal{H}_1)$.
2. The adjoint or dual map $\Phi^* : \mathfrak{B}(\mathcal{H}_2) \to \mathfrak{B}(\mathcal{H}_1)$ is defined via the Hilbert-Schmidt inner product: it is the unique map $\Phi^*$ satisfying $\mathrm{Tr}(B\,\Phi^*(A)) = \mathrm{Tr}(\Phi(B)A)$.
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3. Quantum channels are completely positive trace-preserving linear (CPTP) maps. The dual of a CPTP map is a UCP map.

# Stinespring Dilation Theorem

We present the so-called *ancilla form* of Stinespring's theorem.

## Theorem

*Let* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *be Hilbert spaces. Then for every CP map* $\phi : \mathfrak{B}(\mathcal{H}_1) \to \mathfrak{B}(\mathcal{H}_2)$*, there exists a Hilbert space* $\mathcal{K}$ *and an operator* $V \in \mathfrak{B}(\mathcal{H}_2, \mathcal{H}_1 \otimes \mathcal{K})$ *such that*

$$\phi(X) = V^*(X \otimes I_{\mathfrak{B}(\mathcal{K})})V \text{ for all } X \in \mathfrak{B}(\mathcal{H}_1)$$

*and* $\|\phi(I_{\mathfrak{B}(\mathcal{H}_1)})\| = \|V\|^2$.

# Corresponding Setup in QIT

**Theorem**

*Let* $\Phi : \mathcal{T}(\mathcal{H}_1) \to \mathcal{T}(\mathcal{H}_2)$ *be a CPTP map (quantum channel). Then, for any state* $\rho \in \mathcal{S}(\mathcal{H}_1)$*, there exists a Hilbert space* $\mathcal{K}$*, a pure state* $\psi \in \mathcal{S}(\mathcal{K})$*, and a co-isometry* $V \in \mathfrak{B}(\mathcal{H}_1 \otimes \mathcal{K}, \mathcal{H}_2 \otimes \mathcal{K})$ *such that*

$$\Phi(\rho) = \mathrm{Tr}_{\mathcal{K}}\left(V(\rho \otimes \psi)V^*\right).$$

The operator $\mathrm{Tr}_{\mathcal{K}}$ is the *partial trace* $\mathrm{id}_{\mathfrak{B}(\mathcal{H}_1)} \otimes \mathrm{Tr}(\cdot)$ where Tr is the trace operator in $\mathfrak{B}(\mathcal{K})$.

# Conjugate Channel

Let $\Phi : \mathcal{T}(\mathcal{H}_1) \to \mathcal{T}(\mathcal{H}_2)$ be a quantum channel with Stinespring dilation given by

$$\Phi(\rho) = \mathrm{Tr}_{\mathcal{K}}\left(V(\rho \otimes \psi)V^*\right) \ \forall \rho \in \mathcal{S}(\mathcal{H}_1).$$

The *conjugate* or *complementary* channel is $\Phi^C : \mathcal{T}(\mathcal{H}_1) \to \mathcal{T}(\mathcal{K})$ with Stinespring dilation given by

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**Definition**

A subspace of states $\mathcal{C} \subseteq \mathcal{S}(\mathcal{H})$ is an *error correcting code* for a channel $\mathcal{E}$ if there exists a recovery channel $\mathcal{R}$ such that $\mathcal{R} \circ \mathcal{E}(\rho) = \rho$ for all $\rho \in \mathcal{C}$.

# Private Quantum Channels

Motivating example: Random Unitary Channels

**Definition**

A quantum channel $\Phi : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ is said to be a *random unitary channel* if it can be realized as

$$\Phi(X) = \sum_{i=1}^{d} p_i U_i X U_i^* \quad \forall X \in \mathcal{T}(\mathcal{H}),$$

where $U_i$ are unitary operators and $\{p_i\}_{i=1}^{d}$ is a probability distribution.

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# Private Quantum Channels/Codes

## Definition

A subspace of states $\mathcal{C} \subseteq \mathcal{S}(\mathcal{H})$ is a *private code* or *private subspace* for the channel $\Phi : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{K})$ if there is a fixed output state $\rho_0$ such that

$$\Phi(\rho) = \rho_0 \quad \forall \rho \in \mathcal{C}.$$

The channel $\phi$ is called a *private channel*.

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# Two Sides of a Coin

**Theorem (Kretschmann–Kribs–Spekkens (2008))**

*Given a conjugate pair of CPTP maps $\phi$, $\phi^C$, a code is an error-correcting code for one if and only if it is a private code for the other.*

$$\Phi(\rho) = \mathrm{Tr}_{\mathcal{K}}\left(V(\rho\otimes\psi)V^*\right)$$
$$\Phi^C(\rho) = \mathrm{Tr}_{\mathcal{H}_2}\left(V(\rho\otimes\psi)V^*\right)$$

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# When the Algebraic Bridge Breaks Down

**Definition**

A subspace of states $\mathcal{C} \subseteq \mathcal{S}(\mathcal{H}_A)$ is a *private subsystem* for the channel $\Phi : \mathcal{T}(\mathcal{H}_A) \otimes \mathcal{T}(\mathcal{H}_B) \to \mathcal{T}(\mathcal{K})$ if there is a fixed ancillary state $\rho_{B_0} \in \mathcal{S}(\mathcal{H}_B)$ and fixed output state $\rho_0$ such that

$$\Phi(\rho_A \otimes \rho_{B_0}) = \rho_0 \quad \forall \rho_A \in \mathcal{C}.$$

The channel $\phi$ is called a *private channel*.

In Jochym-O'Connor–Kribs–Laflamme–P. (2013, 2014), we showed that a private subsystem can exist in the absence of a private subspace, and provided an example of a private channel with a private subsystem, whose conjugate channel is also private!

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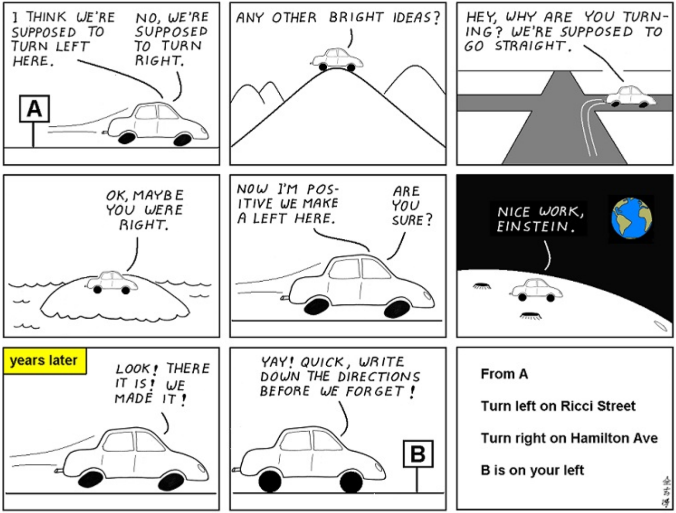
# Mathematical Research vs How Papers are Written




This is how most mathematical proofs are written.

# Factorizable Channels

Recall Stinespring's Dilation Theorem: Let $\Phi : \mathcal{T}(\mathcal{H}_1) \to \mathcal{T}(\mathcal{H}_2)$ be a quantum channel. Then, for any state $\rho \in \mathcal{S}(\mathcal{H}_1)$, we have

$$\Phi(\rho) = \mathrm{id} \otimes \mathrm{Tr}\left(V(\rho \otimes \psi)V^*\right)$$

for some Hilbert space $\mathcal{K}$, pure state $\psi$, and co-isometry $V$.

**Definition (Theorem, Haagerup–Musat, 2011)**

A channel $\phi : M_n \to M_n$ is *factorizable* or has an *exact factorization* if there exists a finite von Neumann algebra $\mathcal{N}$ equipped with a normal faithful tracial state $\tau_N$ and a unitary operator $U \in M_n(N) = M_n(\mathbb{C}) \otimes \mathcal{N}$ such that

$$\phi(\rho) = (\mathrm{id}_{M_n(\mathbb{C})} \otimes \tau_{\mathcal{N}})(U^*(\rho \otimes I_{\mathcal{N}})U), \quad \forall \rho \in M_n(\mathbb{C}).$$

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# Another Result of Haagerup–Musat (2015)

**Definition (Haagerup–Musat, 2015)**

Let $\phi : M_n \to M_n$ be a channel. Then $\phi$ is *factorizable of degree $d$* if $\delta_d \otimes \phi$ is a random unitary channel, where $\delta_d : M_d \to M_d$ is the completely depolarizing channel.

**Theorem**

*A channel is factorizable of degree $d$ if and only if it has an exact factorization through $M_d \otimes \mathcal{N}$, where $\mathcal{N} = M_k \otimes L^\infty[0,1]$.*

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# Schur Multipliers

For any $C \in M_k$, the corresponding *Schur multiplier* is the map $S_C : M_k \to M_k$ given by Schur multiplication: $X \mapsto C \circ X$. The Schur multiplier $S_C$ is a unital quantum channel if and only if $C$ is a *correlation matrix*, i.e., a positive semidefinite matrix whose diagonal elements are all equal to 1.

# A theorem of O'Meara and Pereira (2013)

For any $d, k \in \mathbb{N}$, let

$$\mathcal{F}_k(d) = \{\frac{1}{d}(\mathrm{Tr}(U_i^* U_j))_{i,j=1}^k \in M_k \,:\, U_1, \ldots, U_k \in M_d \text{ unitaries}\}.$$

Since the normalized trace $\frac{1}{d}\mathrm{Tr}\colon M_d \to \mathbb{C}$ is unital and completely positive, it follows that $\mathcal{F}_k(d)$ is a set of correlation matrices.

**Theorem**

*Let $k \in \mathbb{N}$ and let $C \in M_k$ be a correlation matrix. The Schur multiplier $S_C\colon M_k \to M_k$ is a random unitary channel if and only if $C$ lies in the convex hull of the rank-one correlation matrices in $M_k$ (i.e., iff $C$ is in* $\mathrm{conv}(\mathcal{F}_k(1))$*).*

# Connecting O'Meara–Pereira to Factorizable Channels

**Theorem (Harris–Levene–Paulsen–P.–Rahaman, 2018)**

*Let $C \in M_k$ be a correlation matrix and let $d \in \mathbb{N}$. The map $\delta_d \otimes S_C : M_d \otimes M_k \to M_d \otimes M_k$ is a random unitary channel if and only if*

$$C \in \mathrm{conv}(\mathcal{F}_k(d)).$$

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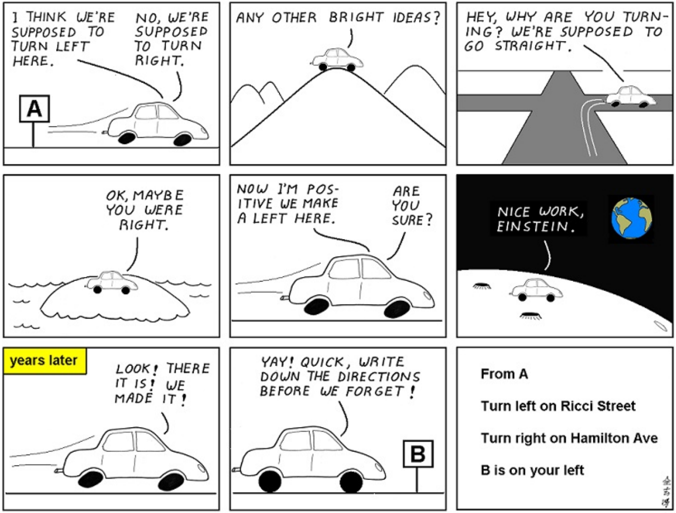


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# Quantum Majorization

**Definition**

Let $\rho_{AB} \in \mathcal{T}(\mathcal{H}_A \otimes \mathcal{H}_B)$ and $\sigma_{AC} \in \mathcal{T}(\mathcal{H}_A \otimes \mathcal{H}_C)$ be two states. We say that $\rho_{AB}$ *quantum majorizes* $\sigma_{AC}$, and write $\sigma_{AC} \prec_q \rho_{AB}$, if there exists a quantum channel $\Phi : \mathcal{T}(\mathcal{H}_B) \to \mathcal{T}(\mathcal{H}_C)$ such that $\mathrm{id} \otimes \Phi(\rho_{AB}) = \sigma_{AC}$.

The conditional min-entropy, $H_{\min}(A|B)_\rho$, of a state $\rho_{AB}$, is defined as

$$H_{\min}(A|B)_\rho := -\log \inf_{\sigma_B \geq 0} \{\operatorname{Tr} \sigma_B : I_A \otimes \sigma_B \geq \rho_{AB}\}.$$

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# A Result of Gour et al, 2018, for Finite Dimensions

**Theorem**

*Let* $\rho_{AB} \in \mathcal{T}(\mathcal{H}_A \otimes \mathcal{H}_B)$ *and* $\sigma_{AC} \in \mathcal{T}(\mathcal{H}_A \otimes \mathcal{H}_C)$ *be states. The following are equivalent:*

1. *The state* $\rho_{AB}$ *quantum majorizes* $\sigma_{AC}$,

$$\sigma_{AC} \prec_q \rho_{AB}.$$

2. *For any quantum channel* $\Phi : \mathcal{T}(\mathcal{H}_A) \to \mathcal{T}(\mathcal{H}_{A'})$, *with* $\dim_{A'} = \dim_C$,

$$H_{\min}(A'|B)_{\Phi\otimes\mathrm{id}(\rho_{AB})} \leq H_{\min}(A'|C)_{\Phi\otimes\mathrm{id}(\sigma_{AC})}$$

# For Infinite Dimensions

Notice that the conditional min-entropy corresponds to the operator space tensor norm

$$H_{min}(A|B)_\rho = -\log \|\rho\|_{\mathcal{T}(\mathcal{H}_B)\widehat{\otimes} B(H_A)}$$

where $\widehat{\otimes}$ is the projective tensor product (Pisier 1993).

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# POVMs

Let $X$ be a locally compact Hausdorff space and $\mathcal{O}(X)$ the $\sigma$-algebra of Borel sets of $X$. In particular, if $X$ is a finite set (endowed the discrete topology), then $\mathcal{O}(X)$ is the power set of $X$.

## Definition

A map $\nu : \mathcal{O}(X) \to \mathfrak{B}(\mathcal{H})_+$ is a *positive operator-valued measure (POVM)* if it is ultraweakly countably additive: for every countable collection $\{E_k\}_{k\in\mathbb{N}} \subseteq \mathcal{O}(X)$ with $E_i \cap E_j = \emptyset$ for $i \neq j$ we have

$$\nu\left(\bigcup_{k\in\mathbb{N}} E_k\right) = \sum_{k\in\mathbb{N}} \nu(E_k)\,,$$

and if $\nu(X) = I_{\mathfrak{B}(\mathcal{H})}$.

# Clean Partial Order

## Definition (Pellonpää 2011)

Let $\nu_1$ and $\nu_2$ be POVMs on $(X, \mathcal{O}(X))$ with values in $\mathfrak{B}(\mathcal{H}_1)$ and $\mathfrak{B}(\mathcal{H}_2)$ respectively.

1. $\nu_1$ is *cleaner than* $\nu_2$ if $\nu_2 = \Phi^* \circ \nu_1$ for some quantum channel $\Phi : \mathcal{T}(\mathcal{H}_2) \to \mathcal{T}(\mathcal{H}_1)$;
2. $\nu_1$ and $\nu_2$ are *cleanly equivalent* if $\nu_1$ and $\mu_2$ are mutually cleaner than each other;
3. $\nu_1$ is *clean* if there is no quantum probability measure cleaner than $\nu_1$, without $\nu_1$ being cleaner than it.

# Approximate Clean Partial Order

Given a quantum channel $\Phi$, the adjoint map $\phi^*$ is *normal*. However, the set of normal maps is not closed in the topology of interest—namely, the point-ultraweak topology—and for this reason we consider *approximately normal unital completely positive linear maps*.

**Definition (Farenick–Floricel–P., 2013)**

Let $\nu_1$ and $\nu_2$ be POVMs.

1. $\nu_1$ is *approximately cleaner* than $\nu_2$ if $\nu_2 = \phi \circ \nu_1$ for some approximately normal ucp map.
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# COIs and Operator Systems

## Definition

An *operator system* is a linear subspace $\mathcal{S}$ of a unital C$^*$-algebra $\mathcal{A}$ such that $\mathcal{S}$ contains the identity and is closed under adjoints.

## Definition

Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems. A linear transformation $\phi : \mathcal{S} \to \mathcal{T}$ is a *unital complete order isomorphism* if $\phi$ is a unital, linear bijection in which both $\phi$ and $\phi^{-1}$ are completely positive.

We characterized approximate cleaner than and approximate cleanness via some norm inequalities and the existence of a complete order isomorphism between operator systems corresponding to the respective quantum probability measures (Farenick–Floricel–P., 2013).

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Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems. A linear transformation $\phi : \mathcal{S} \to \mathcal{T}$ is a *unital complete order isomorphism* if $\phi$ is a unital, linear bijection in which both $\phi$ and $\phi^{-1}$ are completely positive.

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# COIs and Operator Systems

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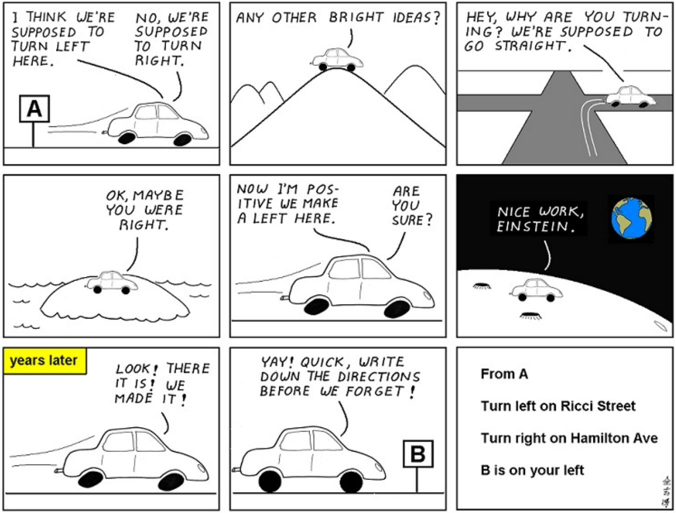
# Mathematical Research vs How Papers are Written




This is how most mathematical proofs are written.

# The End

**Thank you!**