Free Probability and Free Entropy
(A High-Level Introduction)

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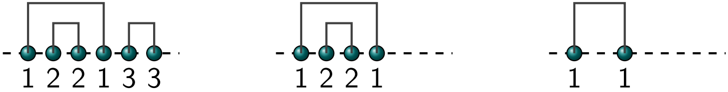

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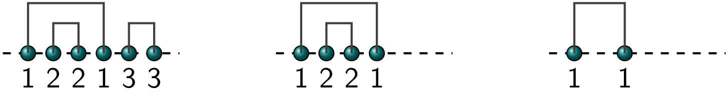

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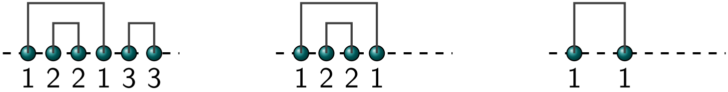

Actually, this occurs if and only if π is non-crossing.

Otherwise, $\phi(\pi) = 0$.

Finishing Up the Proof

At the end of the first lecture, we knew:

$$\lim_{k\to\infty} \phi(S_k^n) = \sum_{\pi\in\mathcal{P}_2(n)} \phi(\pi).$$

For even moments, since only non-crossing partitions π give a non-zero contribution, we have

$$\lim_{k\to\infty} \phi(S_k^{2n}) = \sum_{\pi\in NC_2(2n)} \phi(\pi) = \sigma^{2n} \cdot |NC_2(2n)|.$$

Since $|NC_2(2n)| = C_n$, the nth Catalan number, we are done!

Theorem (Free Central Limit Theorem)

If $(a_i)_{i\in\mathbb{N}}$ are self-adjoint, freely independent, identically distributed nc random variables with $\phi(a_i) = 0$ and $\phi(a_i^2) = \sigma^2$, then

$$\frac{1}{\sqrt{k}}(a_1 + \cdots + a_k) = S_k \to \mathcal{S}(\sigma^2) \text{ in distribution.}$$

Aside: Interpolated Free Group Factors

Definition (Interpolated Free Group Factors)

Let $(\mathcal{M}, \tau)$ be a tracial von Neumann algebra and let $\mathcal{R}$ be a copy of the hyperfinite II_1 factor in $\mathcal{M}$. Also let $\omega = \{X^t \mid t \in T\}$ be a semicircular family such that $\mathcal{R}$ and ω are free.

Then for $1 < r \leq \infty$, we define $L(\mathbb{F}_r)$ as the factor $(\mathcal{R} \cup \{p_t X^t p_t \mid t \in T\})''$, where $p_t \in \mathcal{R}$ are projections satisfying $r = 1 + \sum_{t \in T} \tau(p_t)^2$.

Aside: Compressions/Amplifications

For a II_1 factor $\mathcal{M}$ and $0 < t < 1$, we define the *compression* of $\mathcal{M}$ as

$$\mathcal{M}_t \cong p\mathcal{M}p, \qquad \text{for any } p \in \mathcal{P}(\mathcal{M}),\ \tau(p) = t.$$

We extend this notion to *amplifications of* $\mathcal{M}$ by taking tensors with matrix algebras; for $1 < t < \infty$, if we write $t = n \cdot \ell$, where $0 < \ell < 1$ and $n \in \mathbb{N}$, then we define

$$\mathcal{M}_t \cong p\mathcal{M}p \otimes M_n(\mathbb{C}) \cong M_n(p\mathcal{M}p), \qquad \text{for any } p \in \mathcal{P}(\mathcal{M}),\ \tau(p) = \ell.$$

Definition (Fundamental Group)

The **fundamental group** of a II_1 factor $\mathcal{M}$ is $\{t \in \mathbb{R}_+ \ : \ \mathcal{M}_t \cong \mathcal{M}\}$. It is a multiplicative subgroup of $\mathbb{R}_+$.

Aside: Free Group Factor (FGF) Alternative

Theorem (Dykema '92; Radulescu '94)

One of the following two statements must be true:

1. $L(\mathbb{F}_r) \cong L(\mathbb{F}_s)$ *for all* $1 < r, s \leq \infty$, *and the fundamental group of* $L(\mathbb{F}_r)$ *is* $\mathbb{R}_+$ *for all* $1 < r \leq \infty$.
2. $L(\mathbb{F}_r) \ncong L(\mathbb{F}_s)$ *for all* $1 < r, s \leq \infty$, *and the fundamental group of* $L(\mathbb{F}_r)$ *is* $\{1\}$ *for all* $1 < r \leq \infty$.

Addition Formula

$L(\mathbb{F}_r) * L(\mathbb{F}_s) \cong L(\mathbb{F}_{r+s})$, for $1 < r, s \leq \infty$.

Compression Formula

$L(\mathbb{F}_r)_t \cong L\left(\mathbb{F}\left(1 + \frac{r-1}{t^2}\right)\right)$, for $1 < r \leq \infty$, $0 < t < \infty$.

Microstates Free Entropy

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In classical information theory, entropy measures the amount of “information” or “uncertainty” in a random variable.

Boltzmann’s formula from physics says the entropy of a “macrostate” is obtained by counting how many “microstates” correspond to that “macrostate”.

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Classical entropy of a distribution μ on $\mathbb{R}^n$ with density ρ is given by

$$H(\mu) := -\int_{\mathbb{R}} \rho(x_1, \ldots, x_n) \log \rho(x_1, \ldots, x_n)\ dx_1 \cdots dx_n.$$

Idea of $\chi(X)$

Let $X = (x_1, \ldots, x_n)$ be a tuple of self-adjoint elements of (M, τ).

Our "microstates" will be given by tuples of matrices in $M_N(\mathbb{C})_{sa}$ that approximate the mixed moments of X.

"Counting the number of microstates" will be taking the Lebesgue measure in $M_N(\mathbb{C})_{sa} \cong \mathbb{C}^{N^2} \cong \mathbb{R}^{2N^2}$.

We then do some appropriate normalization and want to take limits as $N \to \infty$, and as the level of approximation improves.

Definition of $\chi(X)$

Let (M, τ) be a tracial W^*-probability space and let $x_1, \dots, x_n$ be an n-tuple of self-adjoint elements in M. The *set of approximating microstates* is:

$$\begin{aligned}&\Gamma(x_1, \dots, x_n; N, r, \epsilon)\\ &:= \{(A_1, \dots, A_n) \in M_N(\mathbb{C})^n_{sa} : |\mathrm{tr}(A_{i_1} \cdots A_{i_k}) - \tau(x_{i_1} \cdots x_{i_k})| \leq \epsilon\\ &\text{for all } 1 \leq i_1, \dots, i_k \leq n, 1 \leq k \leq r\}.\end{aligned}$$

Further define:

$$\chi(x_1, \dots, x_n; r, \epsilon) := \limsup_{N \to \infty} \left(\frac{1}{N^2} \log(\Lambda(\Gamma(x_1, \dots x_n; N, r, \epsilon))) + \frac{n}{2} \log(N) \right).$$

The *(microstates) free entropy* $\chi(x_1, \dots, x_n)$ is

$$\chi(x_1, \dots, x_n) := \lim_{\substack{r \to \infty \\ \epsilon \to 0}} \chi(x_1, \dots, x_n; r, \epsilon).$$

Applications to Operator Algebras

Theorem

Let (M, τ) be a tracial von Neumann algebra generated by self-adjoint $x_1, \ldots, x_n$. Assume that $\chi(x_1, \ldots, x_n) > -\infty$. Then

1. *(Voiculescu '96) M does not have property Γ. In particular, M is a factor.*
   M does not have property Γ if for any bounded sequence (t_k) s.t. $\|[x, t_k]\|_2 \to 0$ for all $x \in M$, we have $\|t_k - \tau(t_k)1\|_2 \to 0$ (every central sequence is trivial).

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   A Cartan subalgebra is a maximal abelian subalgebra whose normalizer generates M.

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   A Cartan subalgebra is a maximal abelian subalgebra whose normalizer generates M.
3. *(Ge '98) M is prime.*
   M is prime if it cannot be decomposed as $M = M_1 \bar{\otimes} M_2$ for II_1 factors M_1, M_2.

Non-Microstates Free Entropy

Motivation

Another measure of the amount of "information" a random variable carries is the classical Fisher information.

Classically, entropy can be recovered through an appropriate integral of Fisher information.

This approach provides a more algebraic flavor and avoids the difficulty of finding the size of microstate spaces.

We will skip the classical formulation on this one!

Non-Commutative Derivatives

Definition

Define the *partial non-commutative derivatives* ∂_i as linear mappings

$$\partial_i : \mathbb{C}\langle X_1, \ldots, X_n\rangle \to \mathbb{C}\langle X_1, \ldots, X_n\rangle \otimes \mathbb{C}\langle X_1, \ldots X_n\rangle \text{ by}$$

$$\partial_i 1 = 0, \qquad \partial_i X_j = \delta_{ij} 1 \otimes 1 \quad \text{for } j = 1, \ldots, n,$$

and by the Leibniz rule:

$$\partial_i(P_1 P_2) = \partial_i(P_1) \cdot 1 \otimes P_2 + P_1 \otimes 1 \cdot \partial_i(P_2) \qquad \text{for } P_1, P_2 \in \mathbb{C}\langle X_1, \ldots, X_n\rangle .$$

Exercise: What is ∂_i on monomials? Compute

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$$\partial_i(X_{i(1)} \cdots X_{i(m)}) = \sum_{k=1}^{m} \delta_{i,i(k)} X_{i(1)} \cdots X_{i(k-1)} \otimes X_{i(k+1)} \cdots X_{i(m)}.$$

∂_i as operators on $L^2(M)$

Recall for $x \in M$ we denote $\|x\|_2^2 = \tau(x^*x)$, and $L^2(M)$ is the closure of M under this 2-norm.

If $x_1, \ldots, x_n \in M_{\text{sa}}$, we can consider the operators ∂_i as derivatives on $\mathbb{C}\langle x_1, \ldots, x_n\rangle \subseteq M$ according to

$$\begin{array}{ccc} \mathbb{C}\langle X_1, \ldots X_n\rangle & \xrightarrow{\partial_i} & \mathbb{C}\langle X_1, \ldots, X_n\rangle \otimes \mathbb{C}\langle X_1, \ldots, X_n\rangle \\ \downarrow{\scriptstyle eval} & & \downarrow{\scriptstyle eval} \\ \mathbb{C}\langle x_1, \ldots x_n\rangle & \longrightarrow & \mathbb{C}\langle x_1, \ldots, x_n\rangle \otimes \mathbb{C}\langle x_1, \ldots, x_n\rangle \end{array}$$

(if the evaluation map is an algebra isomorphism).

Motivation for ξ_i, the conjugate variables

As an operator on $L^2(x_1, \ldots, x_n)$, ∂_i is unbounded, but we want them to be "nice", i.e. closable.
$\implies \partial_i^*$ should be densely defined, i.e. $1 \otimes 1 \in D(\partial_i^*)$.

If $1 \otimes 1 \in D(\partial_i^*)$, then set $\xi_i := \partial_i^*(1 \otimes 1)$. Then we have the following relation:

$$\begin{aligned} \tau(\xi_i P(x_1, \ldots, x_n)) &= \langle \partial_i^*(1 \otimes 1) P(\overline{x}), 1 \rangle \\ &= \langle 1 \otimes 1, \partial_i P(\overline{x}) \rangle \\ &= \tau \otimes \tau(\partial_i P(\overline{x})). \end{aligned}$$

Free Fisher Information

Let $x_1, \ldots, x_n$ be self-adjoint elements of (M, τ).

1. We say $\xi_1, \ldots, \xi_n$ form a **conjugate system** for $x_1, \ldots, x_n$ if for all i, $\xi_i \in L^2(x_1, \ldots, x_n)$, and they satisfy the **conjugate relations:** for all $P \in \mathbb{C}\langle X_1, \ldots, X_n \rangle$,

$$\tau(\xi_i P(x_1, \ldots, x_n)) = \tau \otimes \tau((\partial_i P)(x_1, \ldots, x_n)).$$

2. The **free Fisher information** of $x_1, \ldots, x_n$ is defined by

$$\Phi^*(x_1, \ldots, x_n) = \begin{cases} \sum_{i=1}^n \|\xi_i\|_2^2, & \text{if } \xi_1, \ldots, \xi_n \text{ is a conjugate system for } x_1, \ldots, x_n \\ +\infty, & \text{if no conjugate system exists.} \end{cases}$$

Additivity of Φ^* is Equivalent to Freeness

Theorem

Consider $x_1, \ldots, x_n, y_1, \ldots, y_m$ self-adjoint elements of the same tracial von Neumann algebra (M, τ).
Then $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_m\}$ are free if and only if

$$\Phi^*(x_1, \ldots, x_n, y_1, \ldots, y_m) = \Phi^*(x_1, \ldots, x_n) + \Phi^*(y_1, \ldots, y_m).$$

Proof uses a formulation in terms of cumulants characterizing when $\{\xi_i\}$ is a conjugate system for $\{x_i\}$.

Unification Problem

It was an open question for quite some time whether $\chi(X) = \chi^*(X)$. Since the result $\mathrm{MIP}^* = \mathrm{RE}$ gave a negative answer to the Connes Embedding Problem (this means there exist II_1 factors that do not embed into any ultrapower of $\mathcal{R}$), we now know there are cases where $\chi^*(X) < \chi(X)$.

It is still open whether or not $\chi(X) = \chi^*(X)$ on all $\mathcal{R}^{\mathcal{U}}$-embeddable tracial von Neumann algebras.

It is also open whether or not $\chi(X)$ is a von Neumann algebra invariant, i.e. if $W^*(X) = W^*(Y)$, then is it true that $\chi(X) = \chi(Y)$?

Brief Mention: Other Notions of Dimension

- Hayes' 1-bounded entropy
  - defined via covering numbers of the microstate spaces.
  - known to be a von Neumann algebra invariant.
- Charlesworth and Nelson's free Stein dimension
  - algebraic flavor; defined by taking dimension of an appropriate space of derivations.
  - known to be a $*$-algebra invariant.
- Jekel's free entropy for types in model theory
  - generalizes Hayes' 1-bounded entropy to expressions that involve sup and inf in addition to the trace polynomials.
  - known to be a von Neumann algebra invariant.

Takeaways

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- What is free independence?

- (Vaguely) What is free entropy, and why do people try to study it?

- Free Probability, Operator Algebras, Random Matrices

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- There is more to explore!

Thank you!

Hope you learned something from this! Ask me questions anytime!