Emily Peters

Category Theory in OAS GOALS culminating workshop

7/24/2021

⓪ What is category theory?

① What is a category?

Defn

A category $\mathcal{C}$ has
- objects $ob(\mathcal{C})$
- morphisms $\forall X, Y \in ob(\mathcal{C})$ $Hom(X, Y)$

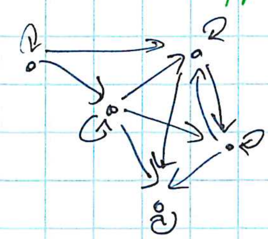
where the interesting stuff happens

which compose:

$\alpha \in Hom(X,Y)$ $\beta \in Hom(Y,Z)$

$\Rightarrow \exists \beta \circ \alpha \in Hom(X,Z)$

and $\forall X \in ob(\mathcal{C})$ $\exists\, id_X \in Hom(X,X)$



EG: SET ob = sets
mor = f'ns

VEC ob = vector spaces
mor = lin trans

HILB ob = Hilb spaces
mor = bndd linear maps

GP ob = gps
mor = homom

G (a group) categorifies ob = •
mor = one ~~~~ arrow for each gen

[NB: Any cat w/ one object, and every morphism is an isomorphism, is a group]

Defn ~~~~ $f: X \to Y$ is isom if $\exists g: Y \to X$ s.t. $f \circ g = id_Y$, $g \circ f = id_X$

$(X \simeq Y)$

Principle of equivalence in cat. theory, never talk about $=$, only Isom.

<u>Defn.</u> A functor is a map of categories: $F: \mathcal{C} \to \mathcal{D}$.
means. $\forall A \in ob(\mathcal{C})$ $F(A) \in ob(\mathcal{D})$
& $\forall A \xrightarrow{f} B$, $F(A) \xrightarrow{F(f)} F(B)$

(... and a map of functors is a natural transformation.)

<u>EG</u>: CAT — ob = small categories
mor = functors

② What do categories do for operator algebras?

$A \subseteq B$ is a subfactor: unital inclusion of factors $\equiv$ VNA w/ trivial centers

<u>Q</u>: Subfactors are a general notion of quantum symmetries; classify them? (in analogy to gp theory)
3 invariants: index
principal graph
standard invariant

Bim (A,B) — ob = A-B bimodules
H Hilb. space ${}_A H_B$
s.t. $a \triangleright h \triangleleft b$ (defined sensibly)
mor = bimodule maps = intertwiners:
bndd linear maps
$f: H \to K$
s.t. $f(a \triangleright h \triangleleft b) = a \triangleright f(h) \triangleleft b$

$\mathrm{Bim}_r(A,B)$ — ob = A-B bimodules that appear as summands of
$X \otimes \bar{X} \otimes X \cdots \otimes X$
where $X = L^2 B$ (completion of B w.r.t. $\langle b_1, b_2\rangle := tr(b_2^* b_1)$)
$\otimes$ = Connes fusion

$\mathrm{Bim}_r(A,A)$, $\mathrm{Bim}_r(B,B)$, $\mathrm{Bim}_r(B,A)$

$\mathrm{Bim}_r(A,A)$ & $\mathrm{Bim}_r(B,B)$ are tensor categories!

<u>Def'n</u> A <u>tensor</u> <u>category</u> $\mathcal{C}$ has some adjectives (abelian k-linear rigid) but most importantly, a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ with an <u>associator</u> $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$

and unit object $\mathbb{1}$ s.t. $\exists\, r_X, l_X$ isomorphisms

$$r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X$$
$$l_X : \mathbb{1} \otimes X \xrightarrow{\sim} X$$

Satisfying lots of axioms

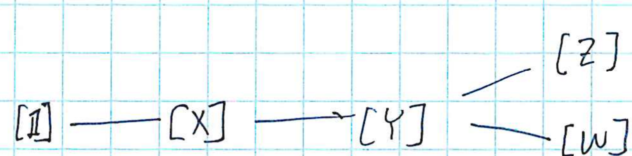
Def'n A <u>simple</u> object in a (k-linear) category is $A \in ob(\mathcal{C})$ such that $Hom(A,A) = k$

Def'n The <u>principal graph</u> of a subfactor has
- a vertex for every isomorphism class of simple objects in $Bim_r(A,A)$ & $Bim_r(B,B)$
- # edges between $\overset{\in}{Y}$, $\overset{\in}{Z}$ $= \dim Hom(Y \otimes X, Z)$

(ie: connect Y to Z if $[Z] \subseteq [Y] \otimes [X]$)

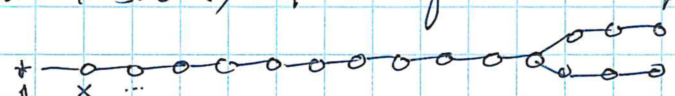
(directed graph? No, because we have Frobenius Reciprocity)

<u>EG</u>:



means, eg, $[X] \otimes [X] = [\mathbb{1}] + [Y]$
$[Y] \otimes [X] = [X] + [Z] + [W]$

<u>Thm</u>: (Asaeda-Yasuda) $\nexists$ subfactor w/ principal graph



<u>Pf</u>: Category theorists proved that all Frobenius-Perron dimensions of objects must be cyclotomic integers.

In the case where the principal graph is finite, $FPdim(X)^2 =$ Jones index!

Jones index of this graph is not cyclotomic integer! $\square$

③ What do OA do for category theory?
<u>Q</u>: [Etingof-Nikshych-Ostrik] Can every finite tensor fusion cat. be defined over a cyclotomic field?
<u>A</u>: [Morrison-Snyder] No! Find a subfactor category (Haagerup, extended Haagerup) which is not.