Constructing subfactors with the jellyfish algorithm

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~~GPOTS
Purdue University
27 May 2015~~

GOALS
summer school
2020

Suppose $N \subset M$ is a subfactor, ie a unital inclusion of type II_1 factors.

Definition

The index of $N \subset M$ is $[M : N] := \dim_N L^2(M)$.

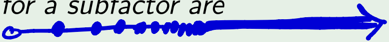
Example

If R is the hyperfinite II_1 factor, and G is a finite group which acts outerly on R, then $R \subset R \rtimes G$ is a subfactor of index $|G|$.

If $H \leq G$, then $R \rtimes H \subset R \rtimes G$ is a subfactor of index $[G : H]$.

Theorem (Jones)

The possible indices for a subfactor are



$$\left\{4\cos\left(\frac{\pi}{n}\right)^2 \middle| n \geq 3\right\} \cup [4, \infty].$$

Let $X = {}_N L^2 M_M$ and $\overline{X} = {}_M L^2 M_N$, and $\otimes = \otimes_N$ or $\otimes_M$ as needed.

Definition

The standard invariant of $N \subset M$ is the (planar) algebra of bimodules generated by X:

$$\begin{array}{ccccccccc} X & , & X \otimes \overline{X} & , & X \otimes \overline{X} \otimes X & , & X \otimes \overline{X} \otimes X \otimes \overline{X} & , & \dots \\ \overline{X} & , & \overline{X} \otimes X & , & \overline{X} \otimes X \otimes \overline{X} & , & \overline{X} \otimes X \otimes \overline{X} \otimes X & , & \dots \end{array}$$

Definition

The principal graph of $N \subset M$ has vertices for (isomorphism classes of) irreducible N-N and N-M bimodules, and an edge from ${}_N Y_N$ to ${}_N Z_M$ if $Z \subset Y \otimes X$ (iff $Y \subset Z \otimes \overline{X}$).

Ditto for the dual principal graph, with M-M and M-N bimodules.

If p.graph is finite,

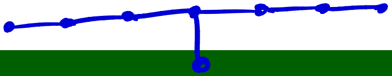
The graph norm of the principal graph of $N \subset M$ is $\sqrt{[M : N]}$.

Example: $R \rtimes H \subset R \rtimes G$

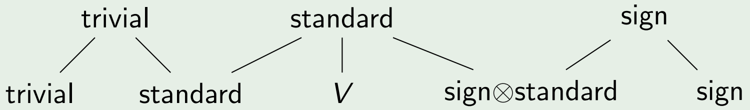
Again, let G be a finite group with subgroup H, and act outerly on R. Consider $N = R \rtimes H \subset R \rtimes G = M$.

The irreducible M-M bimodules are of the form $R \otimes V$ where V is an irreducible G representation. The irreducible M-N bimodules are of the form $R \otimes W$ where W is an H irrep.

The dual principal graph of $N \subset M$ is the induction-restriction graph for irreps of H and G.



Example ($S_3 \leq S_4$)



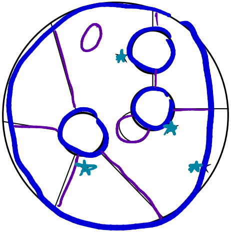
(The principal graph is an induction-restriction graph too, for H and various subgroups of H.)

Planar algebras

Definition (Jones)

A planar diagram has

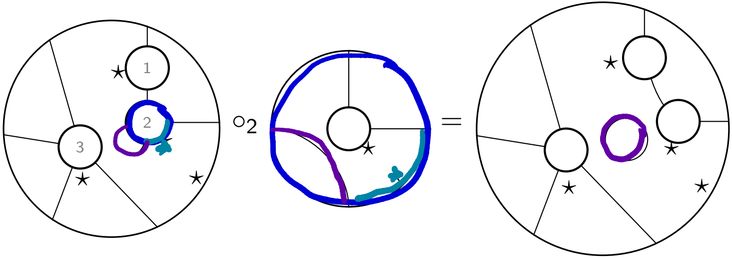
- *a finite number of inner boundary circles*
- *an outer boundary circle*
- *non-intersecting strings*
- *a marked point $\star$ on each boundary circle*



In normal algebra (the kind with sets and functions), we have one dimension of composition:

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

In planar algebras, we have two dimensions of composition



In abstract algebra, sets are given additional structure by functions. For example, a group is a set G with a multiplication law

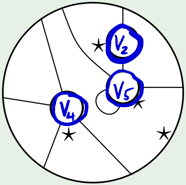
$$\circ : G \times G \to G.$$

A planar algebra also has sets, and maps giving them structure; there are a lot more of them.

Definition

A planar algebra is

- *a family of vector spaces V_k, $k = 0, 1, 2, \ldots$, and*
- *an interpretation of any planar diagram as a multi-linear map*

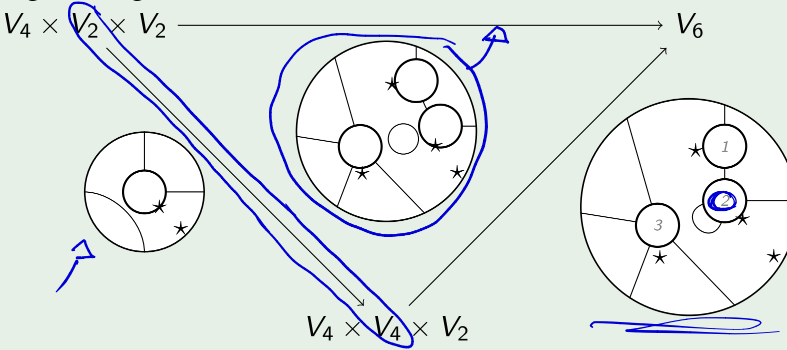
among V_i:  $: V_2 \times V_5 \times V_4 \to V_7$

Definition

A planar algebra is

- *a family of vector spaces* V_k, $k = 0, 1, 2, \ldots$, *and*
- *Planar diagrams giving multi-linear map among* V_i,

such that composition of multilinear maps, and composition of diagrams, agree:



Definition

A Temperley-Lieb diagram is a way of connecting up points on the boundary of a circle labelled $1, \ldots, 2n$*, so that the connecting strings don't cross.*

eg: $n = 4$

For example, when $n = 3$*:*

Example

The Temperley-Lieb planar algebra TL:

- The vector space TL_{2n} has a basis consisting of all Temperley-Lieb diagrams on $2n$ points.
- A planar diagram acts on Temperley-Lieb diagrams by placing the TL diagrams in the input disks, joining strings, and replacing closed loops of string by $\cdot\delta$.

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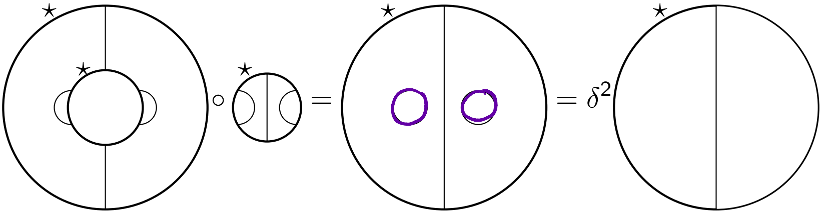
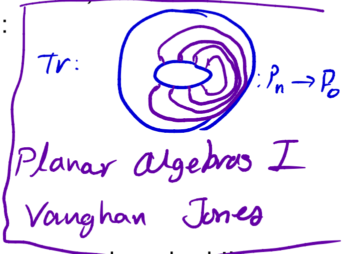
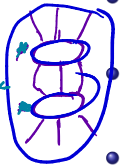


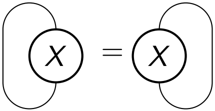
Subfactor planar algebras

The standard invariant of a (finite index, extremal) subfactor is a planar algebra $\mathcal{P}$ with some extra structure:



- $\mathcal{P}_0$ is one-dimensional
- All $\mathcal{P}_k$ are finite-dimensional



- Sphericality: 
- Inner product: each $\mathcal{P}_k$ has an adjoint $*$ such that the bilinear form $\langle x, y \rangle := \mathrm{Tr}(yx^*)$ is positive definite

$P_n \otimes P_n \to P_n$

From these properties, it follows that closed circles count for a multiplicative constant δ.

Definition

A planar algebra with these properties is a subfactor planar algebra.

Theorem (Jones)

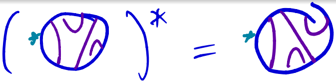
The standard invariant of a subfactor is a subfactor planar algebra.

Theorem (Popa '95)

One can construct a subfactor $N \subset M$ from any subfactor planar algebra $\mathcal{P}$, in such a way that the standard invariant of $N \subset M$ is $\mathcal{P}$ again.

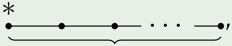
Example

If $\delta > 2$, $TL(\delta)$ is a subfactor planar algebra. If $\delta = 2\cos(\pi/n)$, a quotient of $TL(\delta)$ is a subfactor planar algebra.

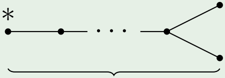


Theorem (Jones, Ocneanu, Kawahigashi, Izumi, Bion-Nadal)

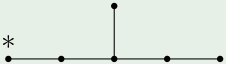
The principal graph of a subfactor of index less than 4 is one of

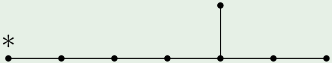
$A_n =$  , $n \geq 2$ — *index* $4\cos^2(\frac{\pi}{n+1})$

n *vertices*

$D_{2n} =$  , $n \geq 2$ — *index* $4\cos^2(\frac{\pi}{4n-2})$

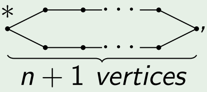
$2n$ *vertices*

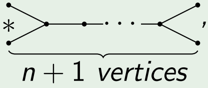
$E_6 =$  — *index* $4\cos^2(\frac{\pi}{12}) \approx 3.73$

$E_8 =$  — *index* $4\cos^2(\frac{\pi}{30}) \approx 3.96$

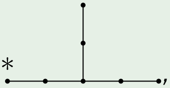
Theorem (Popa)

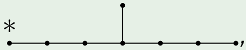
The principal graphs of a subfactor of index 4 are extended Dynkin diagram:

$A_n^{(1)} =$ 

$, n \geq 1,$ $D_n^{(1)} =$ 

$, n \geq 3,$

$E_6^{(1)} =$ 

$,$ $E_7^{(1)} =$ 

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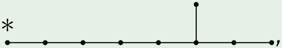
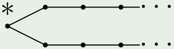
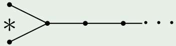
$E_8^{(1)} =$ 

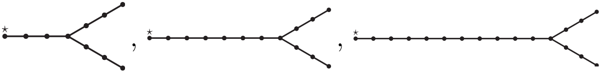
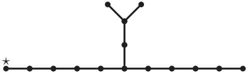
$,$ $A_\infty =$ 

$,$

$A_\infty^{(1)} =$ 

$,$ $D_\infty =$ 

There are multiple subfactors for some of these principal graphs (eg, $n-2$ non-isomorphic hyperfinite subfactors for $D_n^{(1)}$).

- In 1993 Haagerup classified possible finite depth principal graphs for subfactors with index between 4 and $3+\sqrt{3} \approx 4.73$:
  - , . . .,
    (≈ 4.30, 4.37, $4.38, \ldots$)
  - , (≈ 4.56)
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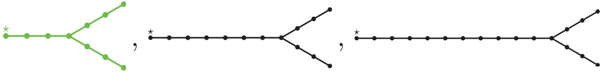
  -  , . . .,
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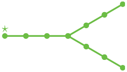
    -  ,  ,  , . . . ,
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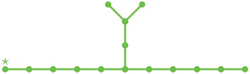
    -  , (≈ 4.56)
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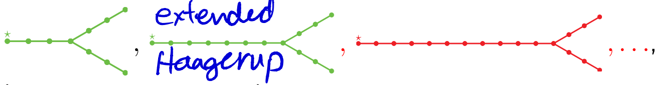
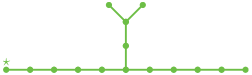
  - 

    $(\approx 4.30, 4.37, 4.38, \ldots)$
  -  , (≈ 4.56)
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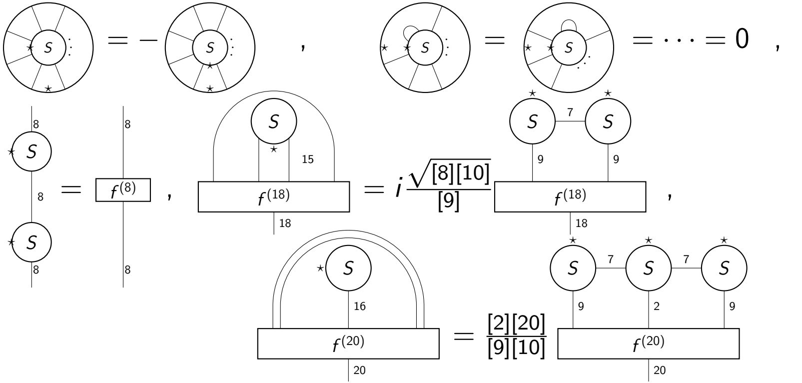
- Bisch (1998) and Asaeda & Yasuda (2007) ruled out infinite families.
- In 2009 we (Bigelow-Morrison-P.-Snyder) constructed the last missing case. `arXiv:0909.4099`

The Extended Haagerup planar algebra

[Bigelow, Morrison, P., Snyder] The extended Haagerup planar algebra is the positive definite planar algebra generated by a single $S \in V_{8,+}$, subject to the relations ◯ $= \delta \approx 4.377$, and



The extended Haagerup planar algebra is a subfactor planar algebra

The Extended Haagerup planar algebra redux

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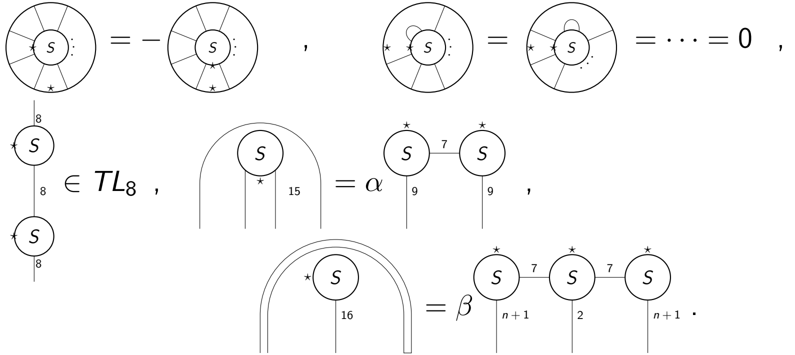


The extended Haagerup planar algebra is a subfactor planar algebra

Let V be the planar algebra generated by this S. To prove V is a subfactor planar algebra: how do we know $V \neq \{0\}$? How do we know $\dim(V_0) = 1$?

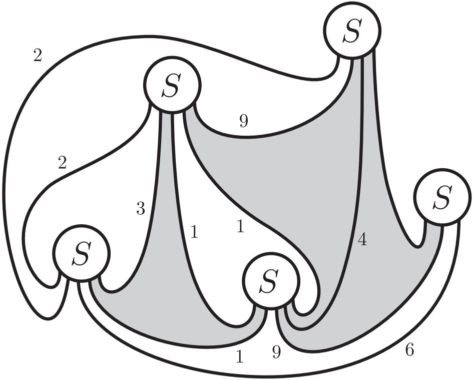
Theorem (Jones-Penneys '10, Morrison-Walker '10)

A planar algebra $\mathcal{P}$ with principal graph Γ is contained in the graph planar algebra $GPA(\Gamma)$.

We show that $V \neq \{0\}$ by finding an element S, satisfying the right relations, in the graph planar algebra of  .

Having $\dim(V_0) = 1$ means we can evaluate any closed diagram as a multiple of the empty diagram. We give an evaluation algorithm, which treats each copy of S as a 'jellyfish' and uses the one-strand and two-strand substitute braiding relations to let each S 'swim' to the top of the diagram.

Begin with arbitrary planar network of Ss.



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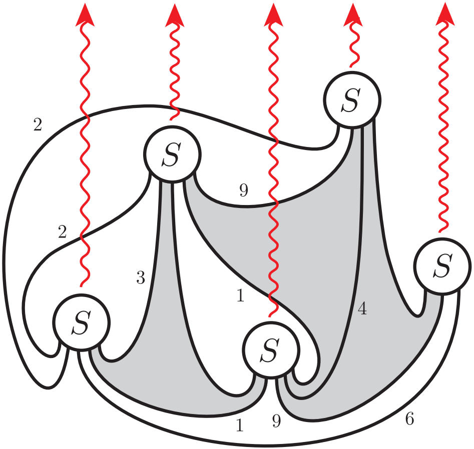

Now float each generator to the surface, using the relation.

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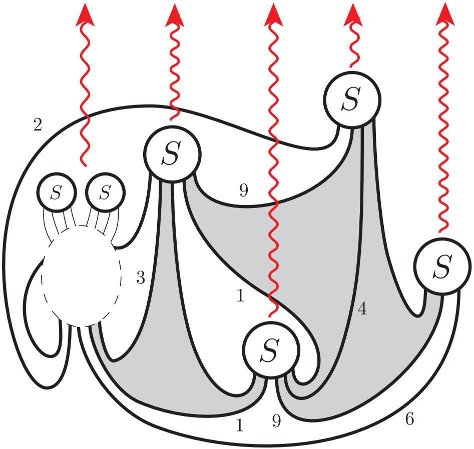

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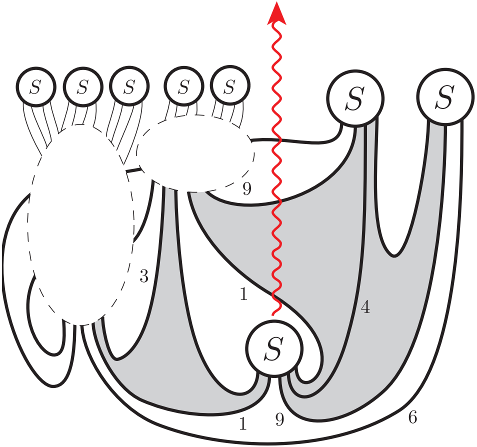

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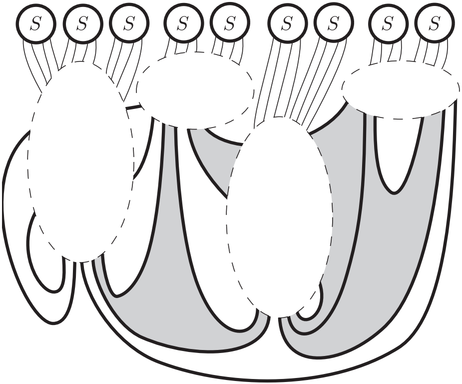
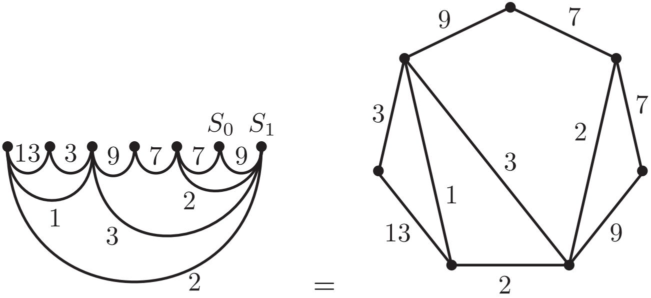


Now float each generator to the surface, using the relation.

The diagram now looks like a polygon with some diagonals, labelled by the numbers of strands connecting generators.



- Each such polygon has a corner, and the generator there is connected to one of its neighbors by at least 8 edges.
- Use $S^2 \in TL$ to reduce the number of generators, and recursively evaluate the entire diagram.

Constructing a subfactor inside its graph planar algebras:

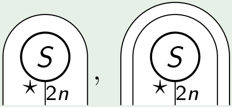
1. Find candidate generators by solving many linear and a few quadratic equations.
2. From principal graph, deduce as many relations as possible; verify that generators found in (1) satisfy these.
3. Find an evaluation algorithm, showing that the relations from (2) make any planar diagram completely evaluable.

(1) and (2) tend to be straightforward/algorithmic, and (3) can be tricky.

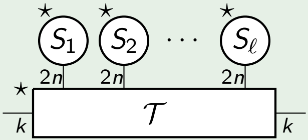
Question: which subfactors have a jellyfish evaluation algorithm?

Definition

A set of (linearly independent, self-adjoint, uncappable, rotational eigenvector) generators $\mathcal{B} \subset \mathcal{P}_{n,+}$ *satisfy jellyfish relations if for each generator* S*, the diagrams*



can be written as linear combinations of trains, which are diagrams



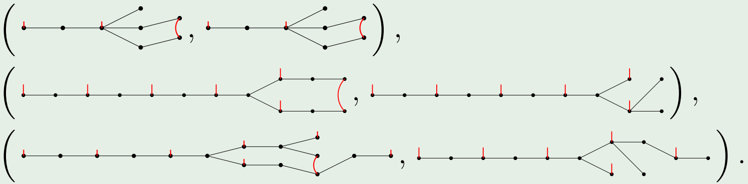
where $S_1, \ldots, S_\ell \in \mathcal{B}$*, and* $\mathcal{T}$ *is a single Temperley-Lieb diagram.*

Theorem (Bigelow-Penneys)

A $n-1$ supertransitive subfactor planar algebra can be constructed using jellyfish generators in $\mathcal{P}_n$ if and only if its principal graph is a spoke graph. We can find 1-strand jellyfish generators if and only if both the principal graph and dual principal graph are spoke graphs.

A spoke graph has a single high-valence hub, with (finite) legs extending out of it. Its supertransitivity is the distance from the first vertex to the hub.

Example

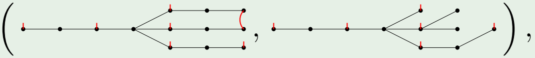


How does one find jellyfish relations?

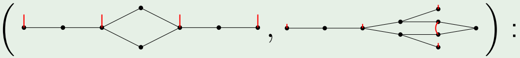
- Acquire the generators in an appropriate graph planar algebra. These generators are an assignment of numbers in a finite extension of $\mathbb{Q}$ to certain loops on a graph.
- Use a computer to evaluate certain closed diagrams with at most 4 generators. This amounts to multiplying rather large matrices, and taking the trace.
- Turn these evaluations of closed diagrams into information about inner products, and then use a computer to derive jellyfish relations for our generators. The use of the computer is limited to basic linear algebra.
- We now have an evaluable planar subalgebra of a graph planar algebra, which is necessarily a subfactor planar algebra. Compute the principal graph by a process of elimination.

Theorem (Penneys-P.)

A subfactor with principal graph $3^{\mathbb{Z}/4}$ *(previously known to Izumi):*



can be found in the graph planar algebra of a different principal graph



The End!