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A is a unital C^*-algebra

Two Qs about C^*-alg:

1. Describe the ideal structure / Is A simple?

2. Describe the tracial structure / Are there any? If yes, is there a unique trace?

- Let Γ be a discrete group, let π be a unitary representation

$\pi : \Gamma \longrightarrow B(H_\pi)$

Take span $\leadsto$ $\mathrm{span}\{\pi(g) : g \in \Gamma\} \subseteq B(H)$ subalgebra which is self adjoint

to make C^* alg $\overline{\mathrm{span}}^{\|\cdot\|}\{\pi(g) : g \in \Gamma\} =: C^*_\pi(\Gamma)$

- <u>Fact</u>: every unital C^*-algebra is of the form $C^*_\pi(\Gamma)$ for some Γ and some π

no canonical choice for π & Γ, tons of them work

make choice of π, Γ then $\Gamma \curvearrowright C^*_\pi(\Gamma)$ action defined by

$$g \cdot a := \pi(g)\, a\, \pi(g)^*$$

<u>Exercise</u>: A state τ on $C^*_\pi(\Gamma)$ is a trace $\Leftrightarrow$ τ is Γ invariant

i.e., $\tau(g \cdot a) = \tau(a)$ $\forall g \in \Gamma$, $a \in C^*_\pi(\Gamma)$

- $\mathbb{X}$ a cmpt space $\Rightarrow$ $C(\mathbb{X})$ unital C^*-alg define

$\Gamma \curvearrowright \mathbb{X}$ $\alpha : \Gamma \longrightarrow \mathrm{Homeo}(\mathbb{X})$ which induces an action

$\Gamma \curvearrowright C(\mathbb{X})$

- Suppose $\varphi : C^*_\pi(\Gamma) \longrightarrow A$ is ucp & A is a Γ-C^*-alg

φ is Γ-equivariant if $\varphi(g \cdot x) = g \cdot \varphi(x)$ $\forall x \in C^*_\pi(\Gamma), g \in \Gamma$

- <u>Fact:</u> If $\varphi : A \to B$ is ucp then $I_\varphi = \{a \in A : \varphi(a^*a) = 0\}$ is a clsd left ideal and $A_\varphi = \left\{ a \in A : \begin{array}{l} \varphi(a^*a) = \varphi(a^*)\varphi(a) \\ \varphi(a a^*) = \varphi(a)\varphi(a^*) \end{array} \right\}$

Mult. domain is largest C^*-subalg st $\varphi|_{A_\varphi}$ is a $*$-homom

Take $a \in A_\varphi$, $b \in A$ $\quad \varphi(ab) = \varphi(a)\varphi(b)$

$\varphi(ba) = \varphi(b)\varphi(a)$

$a \in I_\varphi$

<u>Check:</u> $\varphi(\pi(g)^* a^* a \pi(g)) = g^{-1}\varphi(a^*a) = 0$

$a\pi(g) \in I_\varphi$

- General strategy:

Suppose $\exists$ a Γ-C^*-alg B_Γ st for every Γ-C^*-alg A

$\exists$ Γ-equivariant ucp map $\varphi : A \to B_\Gamma$

Aside:

Category $\mathcal{C}$, object $I \in \mathcal{C}$ is injective $\iff$ $A \subseteq B \in \mathcal{C}$

$\forall$ diagrams
$$\begin{tikzcd} & B \arrow[dr, dashed, "\varphi"] & \\ A \arrow[ur, hook] \arrow[rr] & & I \end{tikzcd}$$
$\exists!$ map φ (uniqueness ?)

Suppose π, Γ are st $\exists!$ $\varphi : C^*_\pi(\Gamma) \to B_\Gamma$ which is faithful $\Rightarrow I_\varphi = 0$

$\Rightarrow$ $C^*_\pi(\Gamma)$ is simple.

Why?

Let $I \trianglelefteq C^*_\pi(\Gamma)$ and let $A := C^*_\pi(\Gamma)/I$ which is a Γ-C^*-alg

$g \cdot \bar{a} = \overline{g \cdot a}$ well-defined (check)

$$j : \begin{tikzcd} C^*_\pi(\Gamma) \arrow[r] \arrow[dr, bend left, "\Gamma\text{- equiv upc map}"] & A \arrow[d] \\ & B_\Gamma \end{tikzcd} \quad \Rightarrow \quad I = 0.$$

$\Lambda \leq \Gamma$ subgroup $\Rightarrow$ $\Gamma \curvearrowright \Gamma/\Lambda$ group action

induces a representation $\lambda_{\Gamma/\Lambda} : \Gamma \to \mathbb{B}(\ell^2(\Gamma/\Lambda))$ $g \in \Gamma$, $\xi \in \ell^2(\Gamma/\Lambda)$

$$(g \cdot \xi)(h\Lambda) = \xi(g^{-1}h\Lambda)$$

If $\Lambda = \Gamma$ then we get the trivial representation

$\Lambda = \{e\}$ gives us the regular representation

$$C^*_{\lambda_{\Gamma/\Lambda}}(\Gamma) = C^*_r(\Gamma)$$

· Thrm (Powers 1976): $C^*_r(\mathbb{F}_n)$ is simple and has unique trace

· Thrm (K.–Scarparo 2021/2022): X cmpt Hausdorff, let $\Gamma \curvearrowright X$ be a boundary action. Let $x \in X$ and $\Gamma_x := \{g \in \Gamma : gx = x\}$. Then $\exists!$ Γ-equiv ucp φ $C^*_{\lambda_{\Gamma/\Gamma_x}}(\Gamma) \to B_\Gamma$ $:= \pi_x$

· Def^n: Let $\Gamma \curvearrowright X$ (cmpt Hausdorff). We say the action is a boundary action iff $\forall x \in X$ $\forall \nu \in \mathrm{Prob}(X)$ $\exists (g_i) \subseteq \Gamma$ st $g_i \cdot \nu \xrightarrow{\text{weak}^*} \delta_x$

$g \cdot \nu(E) = \nu(g^{-1}E)$ Borel probability measures

· Prop: $\exists!$ maximal, universal Γ boundary denoted by $\partial\Gamma$

$\forall$ Γ-boundary X $\exists!$ Γ-equiv cont function $b : \partial\Gamma \to X$

$$\Rightarrow B_\Gamma = C(\partial\Gamma)$$

$$\varphi(\pi_x(g)) = \chi_{\overline{b^{-1}\mathrm{int}(X^g)}}$$

$\leftarrow$ clopen set

$\{x : gx = x\}$