Subfactors and quantum symmetries

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- A *subfactor* is an inclusion of von Neumann algebras $N \subseteq M$ both of which have trivial center (i.e. are factors).
- We will be mostly focused on the case when N, M are II_1 factors (infinite dimensional with normal tracial state).
- Example: If G is a finite group with outer action on a II_1 factor N, and $H \leq G$ we have the subfactors $N \rtimes H \subseteq N \rtimes G$ (See exercise 6.2.3)

Index for subfactors

- Hilbert space representations H of a II_1 factor N are classified by a number $d \in (0, \infty]$ called the *Murray-von Neumann dimension* $\dim_N(H)$ - can take any value in this set.
  - Realize H as subrepresentation of $L^2(N) \otimes \ell^2(\mathbb{N})$.
  - Let P be projection onto H, then $P \in N' \cap B(L^2(N) \otimes \ell^2(\mathbb{N})) \cong N^{op} \otimes B(\ell^2(\mathbb{N}))$
  - $\dim_N(H) := \tau \otimes Tr(P)$.
- The *index* $[M : N]$ of a subfactor $N \subseteq M$ is $\dim_N(L^2(M))$ (for details and alternate descriptions see Section 6.1 of notes)
- $[N \rtimes G : N \rtimes H] = [G : H]$ (the usual group theoretical index).

Remarkable Theorem (V. F. R. Jones, 1983)

The index of a subfactor must lie in the set $\{4\cos^2(\frac{\pi}{n}) \ : \ n \geq 3\} \cup [4, \infty]$, and all possible values are realized by some subfactor.

The standard invariant

- The standard invariant of a finite index subfactor $N \subseteq M$ is an algebraic structure which captures generalized symmetries of inclusion.
- In some situations, a complete invariant! (Will return to this later).
- Reveals deep connections between operator algebras and low dimensional topology and quantum field theory.
- What is it? Many different axiomatizations:
    - Ocneanu: Paragroups (finite depth).
    - Popa: λ-lattices.
    - V.F.R. Jones: Planar algebras.
    - **Longo/Mueger: Algebras in tensor categories.**

Bimodules

- A **bimodule** of a II_1-factor N is a Hilbert space H together with commuting actions of N (left N action) and N^{op} (right N action).
  - $L^2(N)$ ($=$ completion of N w.r.t $\langle n, m\rangle := \tau(nm^*)$), with $n \triangleright m \triangleleft k := nmk$ for $n, m, k \in N$.
  - Let $\alpha \in \mathrm{Aut}(N)$, and define $L^2_\alpha(N) := L^2(N)$ as Hilbert space, with $n \triangleright m \triangleleft k := nm\alpha(k)$ for $n, m, k \in N$.
  - Bimodules generalize automorphisms! (Quantum symmetries!)
- Bimodules form a **category**, whose objects are bimodules and morphisms are *intertwiners*, i.e. if H, K are bimodules over N, an intertwiner is a bounded linear map $f : H \to K$ such that $f(n \triangleright_H \xi \triangleleft_H m) = n \triangleright_K f(\xi) \triangleleft_K m$.
  - Can compose morphisms, associative.
  - Every bimodule has identity morphism.
  - Example: Suppose $\alpha \in Aut(N)$ is inner, i.e. $\alpha(n) = unu^*$. Then define map $L^2_\alpha(N) \to L^2(N)$ by $n \mapsto nu$. (Exercise: check this is bimodule intertwiner)

Relative tensor product

- We said bimodules generalize automorphisms. But automorphisms form a *group*: in particular, you can compose them. Can we "compose" bimodules?
- **Relative tensor product**: **WRONG** but intuitively correct definition: For H, K a pair of $N - N$ bimodules

$$H \boxtimes_N K \text{"} := \text{"} H \otimes K / \langle (\xi \triangleleft n) \otimes \eta - \xi \otimes (n \triangleright \eta) \rangle$$

- Correct definition needs bounded vectors, N-valued inner products, completions etc. (see "Bimodules, Higher Relative Commutants, and Fusion Algebra Associated to a Subfactor" by Dietmar Bisch).
- $L^2_\alpha(N) \boxtimes_N L^2_\beta(N) \cong L^2_{\alpha\circ\beta}(N)$.
- For morphisms $f : H_1 \to H_2$ and $g : K_1 \to K_2$ we can define $f \boxtimes_N g : H_1 \boxtimes_N K_1 \to H_2 \boxtimes_N K_2$.

Rigid C*-tensor categories

The category of bifinite bimodules (left and right Murray von Neumann dimensions are finite) of a II_1 factor N forms a *rigid C*-tensor category* (Think FINITE DIMENSIONAL HILBERT SPACES $\mathbf{Hilb}_{f.d.}$):

- **Semi-simple C*-category** ($\cong \mathbf{Hilb}_{f.d.}^{\oplus n}$, $n \in [1, \infty]$ as a category).
- **Tensor product**: A functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$, $X \times Y \mapsto X \otimes Y$ with
    - **Associator isomorphisms** $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ with coherences (pentagon equations).
    - **Simple unit object** $\mathbb{1} \in \mathcal{C}$, tensoring with $\mathbb{1}$ on left or right is isomorphic to the identity functor (Simple means $\mathrm{End}(\mathbb{1}) \cong \mathbb{C}$)
- **Duals**: Every object X has a "dual" object X^*, and morphisms $ev : X^* \otimes X \to \mathbb{1}$ and $coev : \mathbb{1} \to X \otimes X^*$ morphisms satisfying **duality equations**.
- Rigid C*-tensor categories generalize the category of finite dimensional Hilbert spaces. Objects have a well defined **quantum dimension**, which no longer need be an integer!

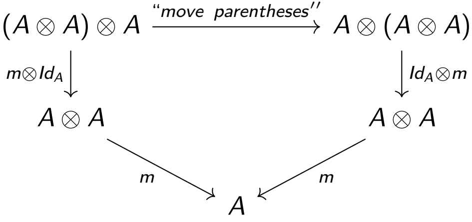
Examples

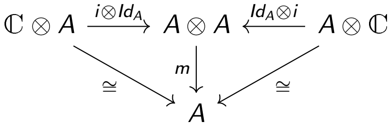
- $\mathrm{Bim}_{b.f.}(N)$ category of bifinite bimodules with tensor product $\boxtimes_N$. The quantum dimension of (an irreducible) bimodule H is $\sqrt{\dim_N(H) \cdot \dim(H)_N}$.
- $\mathrm{Rep}(\mathbb{G})$, where $\mathbb{G}$ is a compact quantum group. Can have non-integer dimensions (e.g. $SU_q(2)$ for $q + q^{-1} \notin \mathbb{Z}$, dimensions of irreps are $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$).
- $\mathcal{C}(\mathfrak{g}, \ell)$ ($\mathfrak{g}$ is simple Lie algebra, $l \in \mathbb{Z}$) coming from conformal field theory, dimensions for $\mathcal{C}(\mathfrak{sl}_2, \ell)$ are $[n]_q$ where q is root of unity related to ℓ.
- "Exotic" examples constructed by hand (using planar algebra techniques).
- Apply constructions to the above list of examples.

We can think of an associative algebra as an object $A \in \text{Vec}$ with

- Multiplication map $m : A \otimes A \to A$.
- Unit map $i : \mathbb{C} \to A$ i.e. $(\lambda \mapsto \lambda 1_A)$

Such that the following diagrams commute:





Makes sense in *any tensor category*: all you need is $\otimes$, moving

Standard invariant revisited

Let $N \subseteq M$ be a finite index inclusion of II_1 factors.

- $L^2(M)$ is a bifinite N-N bimodule.
- Let $\langle L^2(M)\rangle_N \subseteq \mathrm{Bim}_{b.f.}(N)$ be the rigid C*-tensor catgeory generated by $L^2(M)$
- Inclusion $N \subseteq M$ gives a "unit bimodule intertwiner" $L^2(N) \to L^2(M)$. The multiplication on the factor M gives a "multiplication bimodule intertwiner" $L^2(M) \boxtimes_N L^2(M) \to L^2(M)$.
- These make $L^2(M)$ into a (C*)-algebra object in $\langle L^2(M)\rangle_N$.
- The index $[M : N]$ is the quantum dimension of the object $L^2(M)$ in the rigid C*-tensor category $\langle L^2(M)\rangle_N$.

Definition

The standard invariant of $N \subseteq M$ is the pair
(Rigid C*-tensor category $\langle L^2(M)\rangle_N, L^2(M)$ as an algebra object).

Abstract standard invariant

Definition

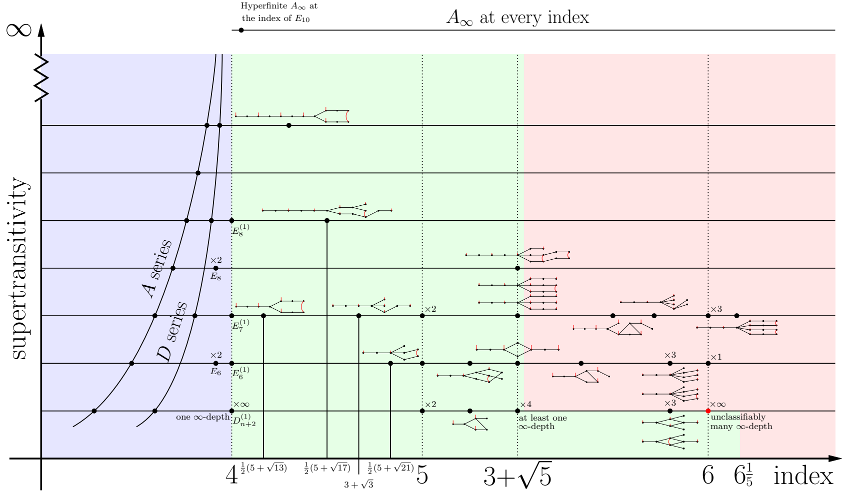
An *abstract standard invariant* is a pair $(\mathcal{C}, A)$, where $\mathcal{C}$ is a rigid C*-tensor category and A is a tensor generating (C*-)algebra object A. The index of the invariant is the quantum dimension of A.

- $\mathcal{C} = Rep(S_n)$, $A = \text{Fun}(\{1, \ldots, n\}, \mathbb{C})$ which is an ordinary commutative associative algebra with poitnwise multiplication. This becomes tensor generating algebra object in $\text{Rep}(S_n)$ from the action S_n on $\{1, \ldots, n\}$.
- More "quantum" example: **Fibb**. Two simple objects $\mathbb{1}, X$ with fusion rules $X^2 = 1 \oplus X$. Then object $1 + X$ has algebra structure, of corresponding subfactor is $\frac{3+\sqrt{5}}{2} = 1 + \frac{1+\sqrt{5}}{2}$.

Question: Can all abstract standard invariants be realized as standard invariants of finite index subfactors? If so in how many ways?

- Popa 1995: Yes! Can all be realized!
- Popa-Shlyakhtenko 2003: Any standard invariant can be realized by a subfactor $N \subseteq M$ with both N, M isomorphic to LF_∞.
- Popa: If $(\mathcal{C}, A)$ is *strongly amenable* then there exists a **unique hyperfinite subfactor** (up to isomorphism) relaizing this standard invariant!
- Therefore, classification of (strongly amenable) standard invariants gives a *complete classification of hyperfinite subfactors* up to a natural equivalence relation!
- Can try to classify subfactors by index: this has been acheived up to $5\frac{1}{4}$ by many hands, including: Afzaly, Asaeda, Bigelow, Bion-Nadal, Bisch, Haagerup, Izumi, V.F.R. Jones, Kawahigashi, Morrison, Ocneanu, Penneys, Peters, Popa, Snyder, Tener ...

Azfaly, Morrison, Penneys:



Some open questions

- There exists (irreducible) subfactor standard invariants with index $d \in [4, \infty)$, but these are nonamenable for index > 4. Which can be realized by a **hyperfinite** subfactor? Is there a gap?
- What are useful invariants for non-amenable subfactors of the hyperfinite (beyond the standard invariant!)
- Given a II_1 N (e.g. a group factor), what can we say about the possible values of the index (or standard invariants) of extensions $N \subseteq M$?
- Classify standard invariants (hence hyperfinite subfactors) by other measures of complexity besides index (Small dimension, Skein theory, Rank, etc.)