GOALS: Cuntz algebras and beyond

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July 23, 2021

Overview

1. Discuss base objects from which we will build C^*-algebras, and relationships between these base objects.
   (a) directed graphs, denoted E
   (b) $\{0,1\}$-matrices, denoted A
   (c) C^*-correspondences, denoted X
2. Define a class of C^*-representations which encode properties of the base objects.
   (a) Cuntz-Krieger (CK) E-families
   (b) Cuntz-Krieger (CK) A-families
   (c) Toeplitz and Toeplitz covariant representations
3. Discuss the universal C^*-algebras of interest for each class of representations, and relations between them.
   (a) graph C^*-algebras, $C^*(E)$
   (b) Cuntz-Krieger algebras, $\mathcal{O}_A$
   (c) Toeplitz-Pimsner algebras, $\mathcal{T}_X$ and Cuntz-Pimsner algebras $\mathcal{O}_X$

1 Base objects

The C^*-algebras we'll discuss in this talk are constructed from base objects including directed graphs, $\{0,1\}$-matrices, and modules. From each base object, one can associate different representations in a C^*-algebra. Then, one can ask questions like: is there a universal C^*-algebra for these families?

1.1 Directed graphs

Let $E = (E^0, E^1, r, s)$ be a directed graph, where

- E^0 is the vertex set
- E^1 is the edge set
- $r : E^1 \to E^0$ is the range map
- $s : E^1 \to E^0$ is the source map

In this talk, we assume E is *row-finite*, which means that for each vertex $v \in E^0$, we have finitely many edges coming out: $|s^{-1}(v)| < \infty$.

Definition 1.1. Given a row-finite graph E, the *line graph* of E, denoted $\hat{E}$, is given by

- $\hat{E}^0 := E^1$

- $\hat{E}^1 := \{ef : e, f \in E^1, r(e) = s(f)\}$
- $s(ef) = f$ for all $ef \in \hat{E}^1$
- $r(ef) = e$ for all $ef \in \hat{E}^1$,

where ef denotes a path of length 2 composed of edges $e, f \in E^1$ such that $s(e) = r(f)$.

1.2 $\{0, 1\}$-matrices

We will be interested in $\{0, 1\}$-matrices that are related to the structure of a finite graph E. Given a finite graph E, we can naturally define two square matrices:

- the *vertex matrix* (aka, *adjacency matrix*) for E is $A = (A_{vw})_{v,w \in E^0}$ where

$$A_{vw} = \begin{cases} 1 & v \text{ is adjacent to } w \\ 0 & \text{else} \end{cases}$$

- the *edge matrix* (aka, line graph's adjacency matrix*) for E is* $\hat{A} = (\hat{A}_{ef})_{e,f \in E^1}$ *where*

$$\hat{A}_{ef} = \begin{cases} 1 & s(e) = r(f) \\ 0 & \textit{else} \end{cases}$$

Given $\{0, 1\}$-matrix B, one can construct a graph E such that B is the vertex matrix for E, but it may not be possible to find a graph for which B is edge matrix.

Remark 1.2. Markov shift spaces are intrinsically linked to $\{0, 1\}$-matrices like A.

2 Graph C^*-algebras

Definition 2.1. A *Cuntz-Krieger E-family* is a collection of mutually orthogonal projections $P := \{P_v : v \in E^0\}$ and partial isometries $S := \{S_e : e \in E^1\}$ acting on a Hilbert space $\mathcal{H}$ which satisfy

1. (CK1) $\quad \forall e \in E^1$: $\quad P_{r(e)} = S_e^* S_e$
2. (CK2) $\quad \forall v \in s(E^1)$: $\quad P_v = \sum_{\{e : s(e) = v\}} S_e S_e^*$

Remark 2.2. Condition (CK2) makes apparent why we require that E is row-finite. Also, some authors (like Raeburn in his book *Graph Algebras*) flip the range and source maps in their definition of (CK1) and (CK2). It's a matter of convention.

Given a Cuntz-Krieger E-family $\{S, P\}$, we can generate a graph C^*-algebra $C^*(S, P)$. You might be asking, "Is this always an interesting object?" Well, no, not if you allow trivialities in your choice of Cuntz-Krieger E-family citation. "For a fixed graph E, do different Cuntz-Krieger E-families produce different C^*-algebras?" They certainly can citation. "So which Cuntz-Krieger E-family is the *best* one to look at?" Funny you should ask! It turns out that the *right* one is constructed in a purely algebraic setting, and we use $\{s, p\}$ to denote this universal Cuntz-Krieger E-family citation.

Definition 2.3. The universal graph C^*-algebra for E, or simply the graph C^*-algebra for E, is the C^*-algebra generated by $\{s, p\}$, and is typically denoted by $C^*(E)$.

Given any other Cuntz-Krieger E-family $\{S, P\}$, there exists a $*$-homomorphism from $C^*(E)$ that surjects onto $C^*(S, P)$ which sends generators to generators (Proposition 1.21, Raeburn). Moreover, $C^*(E)$ is the unique C^*-algebra (up to isomorphism) which has this property (Corollary 1.22, Raeburn).

Example 2.4. Consider the graph E with a single vertex and two loops:

- $E^0 = \{v\}$
- $E^1 = \{e, o\}$,
- $r(e) = s(e) = r(f) = s(f) = v$.

Given $x = (x_1, x_2, ...) \in \ell^2(\mathbb{N})$, define two maps $S_e, S_o : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by

- $S_e(x_1, x_2, x_3, ...) := (0, x_1, 0, x_2, 0, x_3, ...)$
- $S_o(x_1, x_2, x_3, ...) := (x_1, 0, x_2, 0, x_3, 0, ...)$

Note that (this is a fun exercise)

- $S_e^*(x_1, x_2, x_3, ...) := (x_2, x_4, x_6, ...)$
- $S_o^*(x_1, x_2, x_3, ...) := (x_1, x_3, x_5, ...)$.

Both S_e and S_o are isometries with range projections $S_e S_e^* = P_e$ and $S_o S_o^* = P_o$. Thus, $S_e S_e^* + S_o S_o^* = I$, so $C^*(S, P) = \mathcal{O}_2$.

For the above graph, what is $C^*(E)$? Did we already find (a faithful representation) of it? If you want to know if you've already found it via a concrete choice of Cuntz-Krieger E-family $\{S, P\}$, there are two primary litmus tests.

Theorem 2.5 (Gauge Invariant Uniqueness Theorem)**.** *Suppose $\{S, P\}$ is a Cuntz-Krieger E-family in a C^*-algebra B with $P_v \neq 0$ for all $v \in E^0$. If there is a continuosu action $\beta : \mathbb{T} \to Aut(B)$ such that $\beta_z(S_e) = zS_e$ and $\beta_z(P_v) = P_v$ for all $e \in E^1$, $v \in E^0$, then $C^*(E) \cong C^*(S, P)$.*

Like Raeburn says, the above tool is useful when you have concrete information about the C^*-algebra $C^*(S, P)$. There are no conditions about the graph that you need to check.

Theorem 2.6 (Cuntz-Krieger Uniqueness Theorem)**.** *Suppose every cycle of E has an entry (aka, satisfies condition (L)), and $\{S, P\}$ is a Cuntz-Krieger E-family in a C^*-algebra B with $P_v \neq 0$ for all $v \in E^0$. Then $C^*(E) \cong C^*(S, P)$.*

Although the punch line is the same, what's neat about the second theorem is that you don't actually need to know much about $C^*(S, P)$ (just non-triviality of the generators themselves)–the graph holds all the hypothesis data.

Exercise 2.7. Let E be as in Example 2.4. Prove that $C^*(E)$ is $\mathcal{O}_2$.

Exercise 2.8 (Proof of Corollary 2.6, Raeburn)**.** Let E be a finite directed graph.

1. Let $\{s, p\}$ be the universal Cuntz-Krieger E-family. Show that $T_{ef} := s_f s_e s_e^*$ is a partial isometry for all $ef \in \hat{E}^1$.
2. Let $Q_e := s_e s_e^*$ for all $e \in E^1$. Verify that $\{T, Q\}$ to be a Cuntz-Krieger $\hat{E}$-family.
3. Use the Gauge Invariant Uniqueness Theorem to prove $C^*(T, Q) \cong C^*(\hat{E})$.
4. Prove $C^*(E) \cong C^*(\hat{E})$.

3 Cuntz-Krieger algebras

Related to graph C^*-algebras are these doo-dads called Cuntz-Krieger algebras. These actually came first, and then graph C^*-algebras were defined by Enomoto and Watani (1980) and studied as more general objects.

3.1 Definition

Throughout, B is an $n \times n$ $\{0,1\}$-matrix with no identically 0 rows.

Definition 3.1. A Cuntz-Krieger B-family is a collection of partial isometries $\{S_i : 1 \le i \le n\}$ which satisfy

- for all $1 \le i, j \le n$: $\quad (S_i S_i^*) \perp (S_j S_j^*)$ unless $i = j$
- for all $1 \le i \le n$: $\quad S_i^* S_i = \sum_j A_{ij} S_j S_j^*$

We require B to have no identically 0 rows so that the second condition is valid for all $1 \le i \le n$.

Just as before, we can generate a C^*-algebra $C^*(S_i)$ from a Cuntz-Krieger B-family, and similar questions arise. Do different Cuntz-Krieger B-families give me different C^*-algebras? Is there a particular Cuntz-Krieger B-family that's the *right* one to look at? For a multitude of reasons, the answers are "usually, no," and "once you associate a graph to A in a particular way, a property of the graph gives you uniqueness." So long as $S_i \neq 0$ for all $1 \le i \le n$ and a certain condition (I) is satisfiesd (that mimics condition (L) for graphs), $C^*(S_i)$ is *the* Cuntz-Krieger algebra for A, denoted $\mathcal{O}_A$.

Example 3.2. Consider the 2×2 $\{0,1\}$-matrix $A := \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. A Cuntz-Krieger A-family consists of two partial isometries S_1, S_2 which have orthogonal range projections and

$$S_1^* S_1 = \sum_j A_{1j} S_j S_j^* = S_1 S_1^* + S_2 S_2^*$$

$$S_2^* S_2 = \sum_j A_{2j} S_j S_j^* = S_1 S_1^* + S_2 S_2^*$$

Because $S_2^* S_1 = 0$, $S_1^* S_2 = 0$, $S_1 S_1^* S_1 = S_1$, and $S_2 S_2^* S_2 = S_2$, it turns out that $S_1 S_1^* + S_2 S_2^*$ is a unit for $C^*(S_1, S_2)$. So, what we're saying is that S_1 and S_2 are not just partial isometries, they are isometries with orthogonal range projections which sum to the identity.

The C^*-algebra in Example 3.2 belongs to a class of well-studied (and well-liked) C^*-algebras.

Definition 3.3. Fix $n \in \mathbb{N}$. The Cuntz algebra on n generators, denoted $\mathcal{O}_n$, is the universal C^*-algebra generated by n isometries whose range projections sum to the identity.

- $\mathcal{O}_n$ is unital by construction.
- $\mathcal{O}_n$ is universal in the sense that, given any other family $\{T_1, ..., T_n\}$ of isometries whose range projections sum to the identity, there is a $*$-homomorphism $\Phi : \mathcal{O}_n \to C^*(T_i)$ which sends generators to generators.
- $\mathcal{O}_n$ is simple, which means that any (non-trivial) representation of $\mathcal{O}_n$ <u>is</u> $\mathcal{O}_n$.
- $\mathcal{O}_n \ncong \mathcal{O}_m$ if $n \neq m$.
- $\mathcal{O}_\infty$ is a thing. It's purely infinite, but no one who is physically present wants to talk about what that means.

3.2 Graph C^*-algebras and Cuntz-Krieger algebras

Exercise 3.4 (2.8, Raeburn)**.** In this exercise, we explore the correspondence between Cuntz-Krieger algebras and graph C^*-algebras for finite graphs with no sinks and no sources.

1. (matrix $\rightsquigarrow$ graph)

   Fix an $n \times n$ $\{0,1\}$-matrix B. Define a graph E with B as its vertex matrix. If $\{S_i : 1 \le i \le n\}$ is a Cuntz-Krieger B-family, define $Q_i := S_i S_i^*$ and $T_{ij} := S_i S_j S_j^*$ for all $1 \le i, j \le n$.

- Prove that $\{Q, T\}$ is a Cuntz-Krieger $\hat{E}$-family
- Prove that $\mathcal{O}_A \cong C^*(\hat{E})$.
- Use Exercise 2.8 to deduce that $C^*(E) \cong \mathcal{O}_A$. finish

2. (graph $\rightsquigarrow$ matrix) Let E be a finite directed graph with no sinks and no sources, and let $\hat{A}$ be the edge matrix for E.
   - Given a Cuntz-Krieger E-family $\{S, P\}$, show that $\{S_e : e \in E^1\}$ is a Cuntz-Krieger $\hat{A}$-family.
   - Conclude that $C^*(E) \cong \mathcal{O}_A$.

In both cases, the Cuntz-Krieger algebra and graph C^*-algebra coincide. We conclude that, in the finite setting, Cuntz-Krieger algebras are graph C^*-algebras arising from graphs with no sinks and no sources.

There are a lot of neat theorems that relate the structure of a graph E to the algebraic properties of $C^*(E)$. We won't get into this, but it's worth noting that these types of relationships are one of the many reasons C^*-algebraists are so jazzed about these objects. It also has provided a tool for the classification of Markov shift spaces–given a shift space $(,\Sigma)$, one can associate $\{0,1\}$-matrix A. It turns out that the K-theory of $\mathcal{O}_A$ is an invariant for the conjugacy class of (σ, Σ).

3.3 Properties of Cuntz-Krieger algebras

Below we compile, without proof, a list of theorems that relate algebraic properties of $\mathcal{O}_A$ to hypotheses about A (or, equivalently, its associated Markov shift space).

- **Theorem 2.14 - Cuntz-Krieger (1980).** If A is an irreducible matrix, then $\mathcal{O}_A$ is simple.
- **Theorem 2.13 - Cuntz-Krieger (1980).** If A satisfies the infamous condition (I), then $\mathcal{O}_A$ is unique. [Condition (I) is a condition on the Markov shift space which is analogous to condition (L) for graphs]
- If A is finite, then $\mathcal{O}_A$ is unital. (more general theorems hold when A is not finite but satisfies certain conditions)
- In almost any case, $\mathcal{O}_A$ is nuclear.
- Let A be possibly infinite-dimensional. Then $\mathcal{O}_A$ is purely infinite if and only if A satisfies condition (II) (no rows or columns of A are identically 0, and no irreducible block of A is a permutation matrix).

4 Cuntz-Pimsner algebras

Cuntz-Pimsner algebras generalize both Cuntz-Krieger algebras (including Cuntz-Krieger algebras one can define for countably-infinite $\{0,1\}$-matrices), crossed products by $\mathbb{Z}$, and graph C^*-algebras. The base object is a C^**-correspondence.*

4.1 C^*-correspondences

Definition 4.1. Let $\mathcal{A}$ be a C^*-algebra. An $\mathcal{A}$-correspondence is a right $\mathcal{A}$-module X equipped with

- a representation $\phi : \mathcal{A} \to \mathcal{L}(X)$
- a bilinear map $\langle \cdot, \cdot \rangle : X \times X \to \mathcal{A}$ that satisfies

  $$\langle x \cdot a, y \rangle = a^* \langle x, y \rangle, \quad \langle x, y \cdot a \rangle = \langle x, y \rangle a, \quad \langle x, y \rangle^* = \langle y, x \rangle$$

  for all $x, y \in X$, $a \in \mathcal{A}$
- X is complete with respect to the induced norm $\|x\| := \|\langle x, x \rangle\|_{\mathcal{A}}^{1/2}$

Example 4.2. Let $E = (E^0, E^1, r, s)$ be a row-finite directed graph, and consider $C_c(E^1)$ as a right $C_0(E^0)$-module where for each $g \in C_0(E^0)$ and $f \in C_c(E^1)$, we define $f \cdot g \in C_c(E^1)$ by

$$(f \cdot g)(e) = f(e)g(s(e)) \quad \forall e \in E^1.$$

We can, in fact, build a $C_0(E^0)$ correspondence via the following:

- Define $\phi : C_0(E^0) \to \mathcal{C}_\rfloor(\mathcal{E}^\infty)$ by $f \mapsto [\phi(g)f]$, where

$$[\phi(g)f](e) = f(r(e))g(e) \quad \forall e \in E^1$$

- Define $\langle \cdot, \cdot \rangle : C_c(E^1) \times C_c(E^1) \to C_0(E^0)$ by $[\langle g, h \rangle](v) = \sum_{e \in s^{-1}(v)} \overline{g(e)} h(e)$.

Mod out by $\{f \in C_c(E^1) : \langle f, f \rangle = 0\}$, then complete this quotient with respect to the induced norm to get a $C_0(E^0)$-correspondence, $X(E)$.

Definition 4.3. A *Toeplitz representation* of a C^*-correspondence X over $\mathcal{A}$ in a C^*-algebra B is a pair of representations (ψ, π), where $\psi : X \to B$ and $\pi : \mathcal{A} \to B$ which satisfies

- $\psi(x \cdot a) = \psi(x)\pi(a)$
- $\psi(x)^*\psi(y) = \pi(\langle x, y \rangle)$
- $\psi(a \cdot x) = \pi(a)\psi(x)$

for all $a \in \mathcal{A}$ and $x \in X$.

You might want to construct a C^*-algebra which is universal with respect to these conditions. We call this algebra the Toeplitz algebra, $\mathcal{T}_X$, and while it is a universal object, there is a concrete representation for it. It turns out that $\mathcal{T}_X$ is precisely the C^*-algebra generated by the creation operators $\{T_x : x \in X\}$ on the Fock space $\mathcal{F}(X)$.

4.2 Fock space

Given a Hilbert space $\mathcal{H}$, define the *Fock space* for $\mathcal{H}$ by

$$\mathcal{F}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n},$$

where $\mathcal{H}^0$ is $\mathbb{C}$. Note that $\mathcal{F}(\mathcal{H})$ is a Hilbert space:

- For each $n \in \mathbb{N}$, let $\mathcal{H}_n := \mathcal{H}^{\otimes n}$ with inner product $\langle \cdot, \cdot \rangle_n$. Then

$$\langle x_1 \otimes ... \otimes x_n, y_1 \otimes ... \otimes y_n \rangle_n = \langle x_1, y_1 \rangle ... \langle x_n, y_n \rangle .$$

- To form a Hilbert space out of a direct sum of Hilbert spaces, one considers only the subset of vectors

$$X := \left\{ (h_n)_{n=0}^{\infty} \in \bigoplus_{n=0}^{\infty} \mathcal{H}_n : \sum_{n=0}^{\infty} \|h_n\|_n^2 < \infty \right\}$$

- The inner product on X is defined to be $\langle (x_n), (y_n) \rangle = \sum_{n=0}^{\infty} \langle x_n, y_n \rangle_n$.

Fix $n \in \mathbb{N}$. For each $h \in \mathcal{H}$, define $T_h : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n+1}$ by $T_h(x_1 \otimes ... \otimes x_n) = h \otimes x_1 \otimes ... \otimes x_n$. The operator $T_h \in B(\mathcal{F}(\mathcal{H}))$ is called a *creation operator*. The adjoint of T_h is called an *annihilation operator* because $T_h^*(x_1 \otimes ... \otimes x_n) = \langle h, x_1 \rangle x_2 \otimes ... \otimes x_n$ for all $x \in \mathcal{H}^{\otimes n}$.

We can actually build Fock spaces for C^*-correspondences. It's done in an analogous way, but care must be taken in defining the Hilbert $\mathcal{A}$-module structure of $X^{\otimes n}$, as well as the encompassing structure $\mathcal{F}(X)$.

Example 4.4. Let's build the Fock space $\mathcal{F}(X(E))$ for the $C_0(E^0)$-correspondence defined above, but let's do it in the specific case where E is the graph with one vertex, v, and two loops, e_1 and e_2.

- $C_0(E^0) \cong \mathbb{C}$
- $C_c(E^1) \cong \mathbb{C}^2$
- the left and right actions of $\mathbb{C}$ on $\mathbb{C}^2$ are the natural ones: $\lambda(\mu, \nu) = (\lambda\mu, \lambda\nu)$ and $(\mu, \nu)\lambda = (\mu\lambda, \nu\lambda)$.
- the $\mathbb{C}$-valued inner product on $\mathbb{C}^2$ is just the (right-linear) dot product.
- $X(E) = \mathbb{C}^2$ (no need to mod out or complete the space)

For each $n \in \mathbb{N}$, $(\mathbb{C}^2)^{\otimes n} \cong \mathbb{C}^{2n}$, so

$$\mathcal{F}(\mathbb{C}^2) = \bigoplus_{n=0}^{\infty} \mathbb{C}^{2n}$$

Neat! Now, note that $\mathbb{C} = \text{span}\{(1,0),(0,1)\}$, and $T_{(1,0)}(\mu_1 \otimes ... \otimes \mu_n) = (1,0) \otimes \mu_1 \otimes ... \otimes \mu_n$ and $T_{(0,1)}(\mu_1 \otimes ... \otimes \mu_n) = (0,1) \otimes \mu_1 \otimes ... \otimes \mu_n$. Let $T_{\mathbf{1}} := T_{(1,0)}$ and $T_{\mathbf{2}} := T_{(0,1)}$, so $C^*(\{T_\mu : \mu \in \mathbb{C}^2\}) = C^*(T_{\mathbf{1}}, T_{\mathbf{2}})$.

- Check that $T_{\mathbf{1}}$ and $T_{\mathbf{2}}$ are isometries in $B(\mathcal{F}(X(E)))$.
- Show that $T_{\mathbf{1}}T_{\mathbf{1}}^* + T_{\mathbf{2}}T_{\mathbf{2}}^* = 1$.

We may conclude that $\mathcal{T}_X \cong C * (T_{\mathbf{1}}, T_{\mathbf{2}})$ is $\mathcal{O}_2$.

4.3 Cuntz-Pimsner algebra

These are sort of yucky to define in the concrete way, although it is super nice that there is a concrete way to get to them. The algebra we constructed from $X(E)$ in the previous example could be generally described as the C^*-algebra of creation operators on $\mathcal{F}(X(E))$, denoted $\mathcal{T}_{X(E)}$ and called *the Toeplitz algebra for $X(E)$*. The Cuntz-Pimsner algebra for $X(E)$, denoted $\mathcal{O}_{X(E)}$, is a quotient of $\mathcal{T}_{(X(E)}$ by the *Katsura ideal.* This is the most concrete approach to getting at $\mathcal{T}_{X(E)}$ and thus $\mathcal{O}_{X(E)}$, although one may also define $\mathcal{T}_{X(E)}$ as a universal C^*-algebra subject to *Toeplitz covariant representations* of the C^*-correspondence $X(E)$, effectively encoding the behavior of $X(E)$ as a Hilbert $C_0(E^0)$-module. Similarly, $\mathcal{O}_X$ is universal with respect to covariant Toeplitz representations that do "an extra nice thing."

- The last two decades have included the development of successful gauge invariant and Cuntz-Krieger uniqueness theorems for Cuntz-Pimsner algebras
- Given an arbitrary C^*-correspondence X over $\mathcal{A}$, the structure of $\mathcal{T}_X$ is very graph C^*-algebra-like. In particular, if you take words in $\{T_x, T_y^* : x, y \in X\}$, you end up with a structure theorem that says $\mathcal{T}_X$ is generated by elements of the form $T_\mu T_\nu^*$, where μ, ν are tuples with entries from X, just like graph C^*-algebras.

4.4 Cuntz-Pimsner algebras, Cuntz-Krieger algebras, and Cuntz algebras

Cuntz-Pimsner algebras are valuable because they generalize the C^*-algebra constructions discussed in this talk.

Proposition 4.5. *When $X = H$ is just a finite-dimensional Hilbert space with dimension n, $\mathcal{O}_X \cong \mathcal{O}_n$.*

Proposition 4.6. *When $\mathcal{A}$ is a finite-dimensional commutative C^*-algebra, so $\mathcal{A} = C(\Sigma)$, the finitely-generated C^*-correspondences over $\mathcal{A}$ are in one-to-one correspondence with $|\Sigma| \times |\Sigma|$-matrices $A = (A_{ij})_{i,j\in\Sigma}$ with nonnegative integer entries. If X is one such C^*-correspondence over $\mathcal{A}$ that has only $\{0,1\}$-entries, then $\mathcal{O}_X \cong \mathcal{O}_A$.*

Proposition 4.7. *If E is a row-finite directed graph, then $\mathcal{O}_{X(E)} \cong C^*(E)$. In the case of Example* 4.4*, the Katsura ideal is $\{0\}$, so $\mathcal{T}_{X(E)} \cong \mathcal{O}_{X(E)} \cong \mathcal{O}_2$.*