Groundwork for Operator Algebras Lecture Series “at" Michigan State University

Free probability, random matrices, and 1-bounded entropy

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Overview

- I'll give a broad strokes background of (things around) microstates/microstates free entropy dimension.
- Roughly speaking, the idea is to understand a tracial vNa by understanding "how many" finite-dimensional approximations it has.
- This neatly connects to random matrices, and allows one to bring tools from the finite-dimensional world (finite-dimensional analysis, probability, measure theory) that are usually unavailable in the infinite-dimensional setting.

Let X be an essentially bounded $\mathbb{R}$-valued random variable. Its *law*, or distribution, is the probability measure μ_X on $[-\|X\|_\infty, \|X\|_\infty]$ defined by

$$\mathbb{E}(f(X)) = \int f \, d\mu_X$$

for all Borel $f\colon [-\|X\|_\infty, \|X\|_\infty] \to \mathbb{C}$.

Exercise: using Stone-Weierstrass and Riesz representation, if ν is any measure so that

$$\mathbb{E}(p(X)) = \int p \, d\nu$$

for all *polynomials* $p\colon [-\|X\|_\infty, \|X\|_\infty] \to \mathbb{C}$, then $\nu = \mu_X$.

Can do same for a tuple $X = (X_1, \cdots, X_r)$, just replace polynomials of one variable with polynomials of several variables.

Concrete Noncommutative Laws

Fix $r \in \mathbb{N}$. We let $\mathbb{C}\langle T_1, \cdots, T_r\rangle$ be the algebra of NC polynomials in r-variables. Give $\mathbb{C}\langle T_1, \cdots, T_r\rangle$ the unique $*$-structure which makes T_j self-adjoint.

Let (M, τ) be a tracial von Neumann algebra and $x \in M^r_{s.a.}$. The *law of* x is the linear function $\ell_x \colon \mathbb{C}\langle T_1, \cdots, T_r\rangle \to \mathbb{C}$ defined by

$$\ell_x(P) = \tau(P(x)).$$

If $r = 1$, and x is self-adjoint, then

$$\ell_x(P) = \int P\, d\mu_x, \;\; \mu_x = \text{the spectral measure of } x \text{ wrt } \tau$$

Fix $r \in \mathbb{N}$, and $R \in [0, \infty)$. Let $\Sigma_{R,r}$ be the space of all linear $\ell\colon \mathbb{C}\langle T_1, \cdots, T_r\rangle \to \mathbb{C}$ so that

- $\ell(P^*P) \geq 0$ for all $P \in \mathbb{C}\langle T_1, \cdots, T_r\rangle$,
- $\ell(PQ) = \ell(QP)$ for all $P, Q \in \mathbb{C}\langle T_1, \cdots, T_r\rangle$,
- $\ell(1) = 1$,
- $|\ell(T_{j_1} T_{j_2} \cdots T_{j_k})| \leq R^k$ for all $j_1, \cdots, j_k \in \{1, \cdots, r\}$.

Exercise using GNS: $\ell \in \Sigma_{R,r}$ if and only if there is a tracial von Neumann algebra (M, τ) and an $x \in M^r_{s.a.}$ so that $\ell = \ell_x$. Moreover, if $\ell = \ell_x$ then $\|x\|_\infty \leq R$, where

$$\|x\|_\infty = \max_{1 \leq j \leq r} \|x_j\|.$$

We can endow $\Sigma_{R,r}$ with the weak*-topology. So $\ell_n \in \Sigma_{R,r}$ converges to ℓ in $\Sigma_{R,r}$ if and only if $\ell_n(P) \to \ell(P)$ for every $P \in \mathbb{C}\langle T_1, \cdots, T_r \rangle$.

Given $R \in [0, \infty)$, a tracial von Neumann algebra (M, τ) and $x \in M_{s.a.}^r$ with $\|x\|_\infty \leq R$, say that x *has microstates* if

$$\ell_x \in \overline{\bigcup_k \{\ell_A : A \in M_k(\mathbb{C})_{s.a.}^r, \|A\|_\infty \leq R\}}^{wk^*}.$$

Equivalently, there is a sequence $A_n \in M_{k(n)}(\mathbb{C})_{s.a.}^r$ with $\|A_n\|_\infty \leq R$ and $\ell_{A_n} \to \ell_x$.

Exercise: x has microstates if and only if $W^*(x)$ admits a trace-preserving embedding into an ultraproduct of matrices.

Random matrices

Let $X^{(k)} = (X_1^{(k)}, \cdots, X_r^{(k)}) \in M_k(\mathbb{C})^r_{s.a.}$ be a random tuple such that:

- $(X_l^{(k)})_{l=1}^r$ is an independent family,
- for each $1 \leq l \leq r$, $k \in \mathbb{N}$ the random variables

$$\{(X_{l,ii}^{(k)}\}_{i=1}^k \cup \{\sqrt{2}\,\mathrm{Re}(X_{l,ij}^{(k)})\}_{1\leq i<j\leq k} \cup \{\sqrt{2}\,\mathrm{Im}(X_{l,ij}^{(k)})\}_{1\leq i<j\leq k}$$

are iid, Gaussian,with mean zero and variance $\frac{1}{k}$.

This is called the *Gaussian unitary ensemble*, denoted $GUE(k, r)$.

Define the *semicircular distribution* to be the probability measure μ_{sc} on $[-2, 2]$ with

$$d\mu_{sc} = \frac{1}{2\pi}\sqrt{4 - x^2}1_{[-2,2]}\,dx.$$

Voiculescu's asymptotic freeness theorem

Theorem (Voiculescu's asymptotic freeness theorem '98)

Fix $r \in \mathbb{N}$, and let $X^{(k)}$ be the Gaussian unitary ensemble $GUE(k, r)$. Let $s = (s_1, \cdots, s_k)$ be a tuple of free independent NC variables with $\mu_{s_j} = \mu_{sc}$ for all j. Then, almost surely,

$$\ell_{X^{(k)}} \to_{k\to\infty} \ell_s.$$

Microstates Free Entropy Dimension

Motivated by his asymptotic freeness theorem, for a tuple $x \in M^r_{s.a.}$ Voiculescu defined the *microstates free entropy dimension*.

One has $\delta_0(x) \geq 1$ if $W^*(x)$ is diffuse (has no nonzero minimal projections), and $\delta_0(x) > 1$ has lots of structural implications for $W^*(x)$. $\delta_0(x) = r$ if x is an r-tuple of free variables.

A priori, $\delta_0(x)$ is not an invariant of $W^*(x)$: i.e. it is possible there exists $y \in W^*(x)^l_{s.a.}$ with $\delta_0(x) \neq \delta_0(y)$ and $W^*(y) = W^*(x)$.

Implicit in work of Jung '07 and explicitly due to *H*. '18 is the notion of *1-bounded entropy*, denoted $h(M)$, which is an invariant.

In the spirit of talks on the assembly map, UCT,.. I will not give the definition.

Axioms

All von Neumann algebras below are diffuse.

- $h(M) \in \{-\infty\} \cup [0, +\infty]$, and $h(M) \geq 0$ iff M has microstates,
- $h(M) = 0$ if M is injective,
- $h(M) = \infty$ if $M = W^*(x)$ and $\delta_0(x) > 1$, e.g. if $M \cong L(\mathbb{F}_r)$,
- $h(N_1 \vee N_2) \leq h(N_1) + h(N_2)$ if $N_1 \cap N_2$ is diffuse,
- $h(W^*(\mathcal{N}_M(N))) \leq h(N)$ if $N \leq M$, where

$$\mathcal{N}_M(N) = \{u \in \mathcal{U}(M) : uNu^* = N\},$$

this last item is *mildly* false. Properly speaking one needs to replace the "1-bounded entropy of $W^*(\mathcal{N}_M(N))$ in the presence of M".

Sample application

Say that $N \leq M$ is *regular* if $W^*(\mathcal{N}_M(N)) = L(\mathbb{F}_r)$. If $N \leq M$ is regular, then $h(M) \leq h(N)$. So

Theorem (Voiculescu '96)

$L(\mathbb{F}_r)$ does not have a diffuse, regular, injective subalgebra.

E.g. $L(\mathbb{F}_r) \not\cong L^\infty(X) \rtimes G$. These results were later recovered by Ozawa-Popa using Popa's deformation/rigidity theory.

Sample Application II

Theorem (Popa '83, Ge '96)

$L(\mathbb{F}_r)$ is prime. I.e. $L(\mathbb{F}_r) \not\cong M_1 \overline{\otimes} M_2$ with M_j diffuse. In fact, $L(\mathbb{F}_r) \not\cong M_1 \vee M_2$ with M_j diffuse and $[M_1, M_2] = \{0\}$.

- Suppose $M = M_1 \vee M_2$ with M_j diffuse and $[M_1, M_2] = \{0\}$.
- Fix $A_j \leq M_j$ diffuse, abelian. Set $N_j = W^*(N_M(A_j))$.
- So $h(N_j) \leq 0$.
- $N_1 \supseteq A_1 \vee M_2$, $N_2 \supseteq M_1 \vee A_2$. So $N_1 \cap N_2 \supseteq A_1 \vee A_2$ is diffuse.
- $h(M) = h(N_1 \vee N_2) \leq h(N_1) + h(N_2) \leq 0$.

Implicitly used the following **exercise:** if (M, τ) is tracial and $N \leq M$ is diffuse, then M is diffuse.

Other applications

Theorem (Popa '83, Houdayer '15, H-Jekel-Nelson-Sinclair)

$L(\mathbb{Z}) = L(\mathbb{Z}) * 1 \leq L(\mathbb{Z}) * L(\mathbb{F}_{r-1})$ has the absorbing amenability property. I.e. if $N \leq L(\mathbb{F}_r)$ is injective and $N \cap L(\mathbb{Z})$ is diffuse, then $N \leq L(\mathbb{Z})$.

Proof uses: 1-bounded entropy, exponential concentration of measure, external averaging property, Voiculescu's asymptotic freeness theorem.

Other applications: weird normalizers

Given $N \leq M$ with N diffuse, define (Pimnser-Popa '86, Izumi-Longo-Popa '98, Popa '99):

- $q\mathcal{N}_M(N) =$ set of $x \in M$ so that there are $a_1, \cdots, a_k \in M$ with
$$xN \subseteq \sum_j Na_j, \text{ and } Nx \subseteq \sum_j a_j N.$$
- $q^1\mathcal{N}_M(N) =$ set of $x \in M$ so that there are $a_1, \cdots, a_k \in M$ with
$$xN \subseteq \sum_j Na_j.$$
- $\mathcal{N}_M^{wq}(N) =$ set of $u \in \mathcal{U}(M)$ so that $uNu^* \cap N$ is diffuse. (Defined in Popa '06 , Ioana-Peterson-Popa '08 ,Galațan-Popa '17)

For each of this, if Q=the vNa generated by the appropriate normalizer, then $h(Q) \leq h(N)$ (by H.).

The weirdest normalizer

Let A be a $*$-algebra. Say two $*$-reps π, ρ of A are *disjoint* if there are no bounded, A-equivariant, linear $T \colon \mathcal{H}_\pi \to H_\rho$.

Example: For X compact, metrizable, and $\mu \in \mathrm{Prob}(X)$, consider $\pi_\mu \colon C(X) \to B(L^2(X, \mu))$ by multiplication operators.

Exercise: π_μ,π_ν are disjoint if and only if $\mu \perp \nu$.

The weirdest normalizer, II

Exercise: π, ρ are disjoint if and only if there is a sequence $(a_n)_n \in A$ with

- $\max(\|\pi(a_n)\|, \|\rho(a_n)\|) \leq 1$,
- $\pi(a_n) \to 1$ SOT,
- $\rho(a_n) \to 0$ SOT.

Hint: use the double commutant theorem and Kaplansky's density theorem.

For $N \leq M$, and $a \in M$, let $L^2(NaN)$ be the closed linear span in $L^2(M, \tau)$ of $\{xay : x, y \in N\}$.

Let $\mathcal{H}_s(N \leq M)$= all $a \in M$ so that $L^2(NaN)$ is disjoint from $L^2(N \overline{\otimes} N)$ as an N-N bimodule.

This contains $q^1\mathcal{N}_M(N)$, $\mathcal{N}_M^{wq}(N)$.

One last application

Theorem (H.)

$h(W^*(\mathcal{H}_s(N \leq M))) \leq h(N)$.

Uses in a crucial way this previous characterization of disjointness.

Has new applications to nonisomorphism results on Free Araki-Woods factors (Shlyakhtenko '04), Bogoliubov cross products (following work of Houdayer-Shlyakhtenko '11), as well as new indecomposability results on free group factors currently not available by other methods. This also solves a conjecture of Galațan-Popa '17.

Suggestions for further reading:

- “Free random variables. A noncommutative probability approach to free products with applications to random matrices, operator algebras and harmonic analysis on free groups." CRM Monograph Series, Voiculescu-Dykema- Nica,
- “Free entropy", arXiv:0103168, Voiculescu, Bulletin of the London Mathematical Society,
- “Strongly 1-bounded von Neumann algebras", Geometric and Functional Analysis, Jung,
- “A Random matrix approach to absorption in free products", to appear in International Mathematics Research Notices, Jekel-H-Nelson-Sinclair,
- “1-bounded entropy and regularity problems in von Neumann algebras", International Mathematics Research Notices, H.

Thanks for paying attention!