Notes For Free Probability Lectures

Hour 1:

In Probability theory have $(\Omega, \mathcal{F}, P)$ & random variables (measurable functions) X

Expectation value $\mathbb{E}$: $\mathbb{E}(X) = \int_\Omega X\,dP$

C^* & W^*-algebras abstractify this notion.

Def: • A *-probability space $(\mathcal{A}, \varphi)$ consists of a *-algebra, $\mathcal{A}$ & a state φ ($\varphi(1)=1$, $\varphi(x^*x) \geq 0\ \forall x$)

• A C^*-prob space is a *-prob space where $\mathcal{A}$ is a C^*-alg (φ is automatically norm continuous)

• A W^*-prob space is where $\mathcal{A}$ is a vN-alg & φ is normal. } often want φ faithful, but not a requirement.

Ex: ① $[L^\infty(\Omega, \mathcal{F}, P), \mathbb{E}]$ (W^*)

② Set $L^{\infty-}(\Omega, \mathcal{F}, P) = \bigcap_{p>1} L^p(\Omega, \mathcal{F}, P)$. Have $(L^{\infty-}(\Omega, \mathcal{F}, P), \mathbb{E})$ & $(M_n(L^{\infty-}(\Omega, \mathcal{F}, P)), \mathbb{E}\circ \mathrm{Tr})$

③ If Γ is a discrete grap, $(C^*_r(\Gamma), tr)$ & $(L(\Gamma), tr)$.

$a_1,\dots,a_n \in (A,\varphi)$ & $a=(a_1,\dots,a_n)$

Universal *-alg

***-distributions:** If a as above, the *-dist of a is a linear functional $\mu_a : \mathbb{C}\langle X_1,X_1^*,\dots,X_n,X_n^*\rangle \to \mathbb{C}$ given by
$\mu_a(p(X_1,X_1^*,\dots,X_n,X_n^*)) = \varphi(p(a_1,a_1^*,\dots,a_n,a_n^*))$

$$\begin{tikzcd} \mathbb{C}\langle X, X^*\rangle \arrow[rr, "X\to a,\ X^*\to a^*"] \arrow[dr, "\mu"'] & & A \arrow[dl, "\varphi"] \\ & \mathbb{C} & \end{tikzcd}$$

If A is a C^* or W^* prob space & $a\in A$ is normal (a normal element), then law of a is given by a measure, ν, supported on $\sigma(a)$. i.e.
$$\mu_a(p(X,X^*)) = \varphi(p(a,a^*)) = \int_{\sigma(a)} p(z,\bar z)\,d\mu$$

Ex: In $(M_n(\mathbb{C}),\mathrm{tr})$, if A is normal, μ_A is the measure on $\mathbb{C}$: $\mu_A = \frac{1}{n}\sum_{j=1}^{n}\delta_{\lambda_j}$

Ex: Haar Unitary: $\varphi(u^k) = \begin{cases} 0 & k\neq 0 \\ 1 & k=0 \end{cases}$ (note $u^* = u^{-1}$ here)

Know $\sigma(u)\subseteq S^1$ so looking μ supported inside S^1 s.t. $\int_{S^1} z^k\,d\mu = 0 \ \forall k\in\mathbb{Z}\setminus\{0\}$.

A: $\mu = \frac{dt}{2\pi}$ (normalized arc length measure) i.e. $\mu(p) = \frac{1}{2\pi}\int_0^{2\pi} p(e^{int}, e^{-int})\,dt$.

Important Example: Full Fock space. $\mathcal{H}$ = Hilbert Space. Form $\mathcal{F}(\mathcal{H}) = \mathbb{C}\Omega \oplus \bigoplus_{n\geq 1}\mathcal{H}^{\otimes n}$ (Ω: "vacuum vector" $\|\Omega\|=1$)

Fix $\xi\in\mathcal{H}$, ($\|\xi\|=1$) define $\ell(\xi)$ by: $\ell(\xi)(\xi_1\otimes\cdots\otimes\xi_n) = \xi\otimes\xi_1\otimes\cdots\otimes\xi_n$ & $\ell(\xi)\Omega = \xi$.

then $\|\ell(\xi)\| = \|\xi\|$ & $\ell(\xi)^*(\xi_1\otimes\cdots\otimes\xi_n) = \langle \xi_1,\xi\rangle\,\xi_2\otimes\cdots\otimes\xi_n$ & $\ell(\xi)^*\Omega = 0$. $\ell(\xi)^*\ell(\xi) = \|\xi\|1$

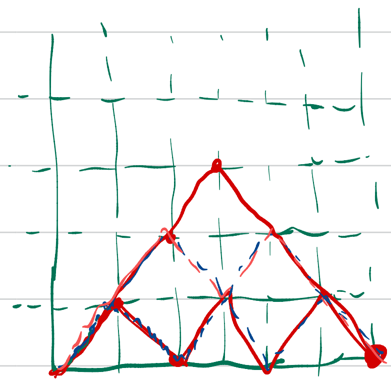
define φ_Ω on $B(\mathcal{F}(\mathcal{H}))$ by $\varphi_\Omega(X) = \langle X\Omega, \Omega \rangle$. Fix $\xi \in \mathcal{H}$, $\|\xi\| = 1$. Find law of $\ell(\xi) + \ell(\xi)^*$.

Compute: $\varphi_\Omega([\ell(\xi)+\ell(\xi)^*]^n) = \langle (\ell(\xi)+\ell(\xi)^*)^n \Omega, \Omega \rangle$: Note $(\ell(\xi)+\ell(\xi)^*)^n = \sum_{i_j \in \{1,*\}} \ell(\xi)^{i_1} \cdots \ell(\xi)^{i_n}$

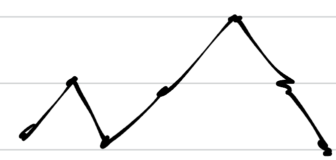
terms that contribute: terms s.t. $\#\{i_k = 1\} = \#\{i_k = *\}$ (forces n even)

and: $\forall j$, $\#\{k \geq j \mid i_k = 1\} \geq \#\{k \geq j \mid i_k = *\}$

If $n = 2k$, these terms are in bijection with Dyck paths:



Proof by picture!



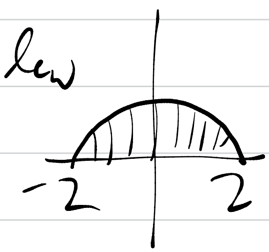
corresponds to

$\ell^*(\xi)\ell(\xi)\ell^*(\xi)\ell^*(\xi)\ell(\xi)\ell(\xi)$

These are counted by the Catalan #'s: $C_n = \frac{1}{n+1}\binom{2n}{n}$

Fact: $\frac{1}{2\pi}\int_{-2}^{2} t^{2n}\sqrt{4-t^2}\,dt = \frac{1}{n+1}\binom{2n}{n}$

So dist is semicircular law



$(\ast, C^*_0 W^*)$

Free Independence: We say unital $A_i \subseteq (A, e)$ for $i \in I$ are $\underline{\ast\text{-free}}$ if $e(a_{i_1} \cdots a_{i_n}) = 0$ whenever: $\underline{a_{i_j} \in A_{i_j}}$, $\underline{i_j \neq i_{j+1}\ \forall j \in \{1, \ldots, n-1\}}$, $\underline{e(a_{i_j}) = 0\ \forall j}$.

i.e. <u>alternating products of centered elements are centered</u>.

- Say sets $\{S_i\}_{i \in I}$ are free if $\{A_i\}_{i \in I}$ are $\ast$-free where A_i the (C^* or W^*) alg gen by S_i
- Say $\{a_i\}_{i \in I}$ is a $\ast$-free family if $\{A_i\}_{i \in I}$ are $\ast$-free whenever A_i is generated by a_i.

Ex: Suppose a & b are free in (A, e) find $e(ab)$

Solution: Note by freeness, $0 = e([a - e(a)1][b - e(b)1]) = e(ab) - e(a)e(b) - e(a)e(b) + e(a)e(b)$.

Thus, $\boxed{e(ab) = e(a)e(b)}$

*By centering, $e(a_{i_1} \cdots a_{i_n})$ can be determined by simply knowing $e(a_i)$.

Ex: If $\Gamma = \Gamma_1 * \Gamma_2$ discrete grps, then $L(\Gamma_1)$ & $L(\Gamma_2)$ are $\ast$-free in $(L(\Gamma), tr)$.

Why: enough to show $\mathbb{C}(\Gamma_1)$ & $\mathbb{C}(\Gamma_2)$ $\ast$-free. If $x \in \mathbb{C}(\Gamma_i)$ has $tr(x) = 0$ & $x = \sum x_g u_g$ then $x_e = 0$ thus suffices to show $tr(u_{g_1} u_{g_2} \cdots u_{g_n}) = 0$ if g_i & g_{i+1} are from different Γ_i.

This is automatic since $g_1 g_2 \cdots g_n \neq e$

Ex: If $\{\xi_i\}_{i\in I}$ is an orthonormal set in $\mathcal{H}$, then $\{\{\ell(\xi_i)\}\}_{i\in I}$ is $*$-free in $(\mathcal{B}(\mathcal{F}(\mathcal{H})), \varphi_\Omega)$

Why: note $\ell(\xi_i)^*\ell(\xi_i)=1$ thus every element in $C^*(\ell(\xi_i),\ell(\xi_i)^*)$ is of the form $\sum_{n,m\geq 0} a_{n,m}(\ell(\xi_i))^n(\ell(\xi_i)^*)^m$. & element is centered $\iff a_{0,0}=0$.

This shows that if $x_{i_k}=\ell(\xi_{i_k})^{n_k}(\ell(\xi_{i_k})^*)^{m_k}$ & $i_k\neq i_{k+1}$, then $\varphi_\Omega(x_{i_1}\cdots x_{i_n})=0$

By induction (using $\ell(\xi_i)^*\ell(\xi_{i'})=0$ for $i,i'\in I$, $i\neq i'$) $x_{i_1}\cdots x_{i_n}\Omega\neq 0 \iff m_k=0\ \forall k$. This means $x_{i_1}\cdots x_{i_n}\Omega=\xi_{i_1}^{\otimes n_1}\otimes\cdots\otimes\xi_{i_n}^{\otimes n_n}$ so $\varphi_\Omega(x_{i_1}\cdots x_{i_n})=0$