GOALS: C^*-envelopes

① <u>Disk algebra</u> $\mathbb{D}$ = open unit disk in $\mathbb{C}$
$\mathbb{T}$ = unit circle

Let $A(\mathbb{D}) = \{ f : \overline{\mathbb{D}} \to \mathbb{C} : f \text{ cont}, f|_{\mathbb{D}} \text{ analytic} \}$

~~What C^*-algebra does A(~~

<u>Note</u>: - $A(\mathbb{D})$ is a unital algebra
- closed under $\|\cdot\|_\infty$ norm
- <u>not</u> a C^*-algebra, $f \mapsto f^* = \bar{f}$ not analytic}

Q. What C^*-algebra does $A(\mathbb{D})$ generate?
A. Depends on where we view it.

<u>One perspective</u>: $A(\mathbb{D}) \subseteq C(\overline{\mathbb{D}})$
and $C^*(A(\mathbb{D})) = C(\overline{\mathbb{D}})$
(weierstrass approx.)

<u>Another perspective</u>: $f \in A(\mathbb{D})$

$$\|f\|_\infty = \sup\{ |f(x)| : x \in \overline{\mathbb{D}} \}$$

$$= \max\{ |f(x)| : x \in \overline{\mathbb{D}} \}$$

max mod principle since analytic

$$= \max\{ |f(x)| : x \in \mathbb{T} \}$$

Hence map $\iota : A(\mathbb{D}) \to C(\mathbb{T})$
$f \mapsto f|_{\mathbb{T}}$

is a (completely) isometric isomorphism.

$$C^*(\mathcal{C}_c(A(\mathbb{D}))) = C(\mathbb{T})$$

$$C(\mathbb{T}) \not\cong C(\overline{\mathbb{D}})$$

Further: $\mathbb{T}$ is smallest set $B \subseteq \overline{\mathbb{D}}$ s.t.

$$f \mapsto f|_B$$

is a complete isometry on $A(\mathbb{D})$

What is the right answer? Best answer?

(For this talk) $C(\mathbb{T})$ best answer

Justification: Let $H = \ell^2(\mathbb{Z})$ hilbert space w/ onb. $(e_n)_{n\in\mathbb{Z}}$

$U e_n = e_{n+1}$ bilateral shift

Then $\sigma(U) = \mathbb{T}$

and $\mathrm{Alg}(I, U) \cong A(\mathbb{D})$

So smallest C^*-alg has recovered the spectrum.

1) Šilov Boundary

X compact Hausdorff space

($C(X)$ unital abelian C^*-alg)

$A \subseteq C(X)$ unital linear subspace

$B \subseteq C(X)$ boundary for A if for each $f \in A$ $\exists\, x \in B$ s.t.

$$|f(x)| = \|f\|_\infty$$

(i.e. $f \mapsto f|_B$ isometric on A)

The unique smallest closed boundary for A is the Šilov Boundary of A.

(exists if A separates points)

So: $\mathbb{T}$ is Šilov boundary for $A(\mathbb{D})$

In terms of C^*-algebras: $A \subseteq C(X)$ unital, linear subspace, separating pts

then $C^*(A) = C(X)$ (Weierstrass)

$B \subseteq X$ closed boundary

$$\iota_B : A \to C(B)$$
$$f \mapsto f|_B$$

completely isometric and $C^*(\iota_B(A)) = C(B)$

S Šilov boundary for A in X

Then $S \subseteq B$ B closed boundary

$\Rightarrow \exists\ J_S \trianglelefteq C(B)$ so that

$$C(B)/J_S \cong C(S)$$

where quotient map completely isometric on A.

$C(S)$ "smallest" C^*-alg. generated by A.

3 Noncommutative Šilov Boundary

C unital C^*-alg, $A \subseteq C$ ~~unital~~ closed alg.
$C^*(A) = C$

The largest ideal $J \trianglelefteq C$ s.t.

$$q : C \longrightarrow C/J$$

is completely isometric on A is the Šilov boundary for A in C.

$(\therefore A \simeq q(A))$

<u>Theorem</u> (Arveson) C/J is independent of C
i.e. if C' C^*-alg.
$\iota : A \to C'$ completely isometric homomorphism
$C^*(\iota(A)) = C'$
If J' Šilov boundary for A in C' then

$$C'/J' \cong C/J$$

<u>Def</u> C/J is the C^*-envelope for A
$C^*_{env}(A)$

~~Example~~ $A_n \subseteq \mathcal{T}_n$ & $\mathcal{O}_n$

Toeplitz   Cuntz

Arveson's Approach $A \subset C$

: let π be an irreducible rep of C
then $\pi|_A$ completely positive map.

π is a boundary rep for A if $\pi|_A$ has a unique completely positive linear extension to C.

Abelian case $A \subseteq C(X)$.
Irreducible reps of $C(X)$ is pt. evaluation
Boundary reps of A are points with unique representing measures (Riesz)

These are called the Choquet Boundary. $Ch(A)$

$Ch(A)$ a boundary
$\overline{Ch(A)}$ = Šilov boundary

Theorem: $J \trianglelefteq$ Boundary ideal ω boundary rep
$J \subseteq \ker \omega$

Arveson's hope: $\bigcap_{\omega \text{ bound}} \ker \omega$ = Šilov boundary

Realized: Davidson-Kennedy (2015)
Arveson (2008)
(after Dritschel-McCullough (2005))

Nagy
Dilation

3 Hamana's Approach: Injectivity.

Theorem (Hamana 1977) Šilov boundary exists / C^*-envelope exists

Why? Injectivity

Roughly: An object C is injective if (the right) maps into C can be extended

Example: Hahn-Banach Theorem $V \subseteq W$ Banach space

$\phi : V \to \mathbb{C}$ bounded linear

$\Rightarrow \exists\ \Phi : W \to \mathbb{C}$ s.t. $\cancel{\|\phi\| = \|\Phi\|}$ $\Phi|_V = \phi$

So: $\mathbb{C}$ is injective in category of Banach spaces and bounded linear maps!

(can assume positive extends to positive)

Example: Arveson Extension Theorem $\mathcal{C}$ C^*-alg

$M \subseteq C$ closed linear self-adjoint, unital (op. sys)

$\phi : M \to B(H)$ completely positive

$\Rightarrow \exists\ \psi : \mathcal{C} \to B(H)$ completely positive

$\psi|_M = \phi$

Hamana Approach

$S \subseteq B(H)$ op. system
(if start with op alg. A let $S = A + A^*$)

(T, K) an injective extension
$K: S \to T$ complete isometric isomorphism
s.t. T injective

(T, K) essential extension if
R another op sys.
$\phi: T \to R$ complete order inj.
if $\phi \circ K$ is.
(i.e. (T, K))

(T, K) injective envelope if it is injective and essential

Hamana Proves: Injective envelope (T, K) for S exists

Then $C^*(K(S))$ is the C^*-envelope of S

Aside Injectivity has played an important rôle in C^*-alg in recent years

Hamana showed (1985): $G \curvearrowright_\alpha A$ group action on C^*

Can be extended $G \curvearrowright_\beta \tilde{A}$ where $A \subseteq \tilde{A}$ and $\tilde{A}$ injective with respect to maps intertwining G-action

Kalantar-Kennedy (2017) Look at $G \curvearrowright \mathbb{C}$ (trivial)

$$\begin{array}{c} G \curvearrowright \mathbb{C} \\ \big\downarrow \text{Hamana} \\ G \curvearrowright \tilde{\mathbb{C}} = C(X) \end{array}$$

Thm $C^*(G)$ simple iff $G \curvearrowright X$ minimal.

4 Semicrossed Products

X compact Hausdorff, $\phi : X \to X$ continuous
(X, ϕ) a dynamical system
(with semigroup $\mathbb{Z}_+$)

Define $\sigma : C(X) \to C(X)$ endomorphism
by $(\sigma(f))(x) = f(\phi(x))$

$(C(X), \sigma, \mathbb{Z}_+)$ C^*-dynamical system

Semicrossed product: $C(X) \times_\sigma \mathbb{Z}_+$ is universal alg. generated by $C(X)$ and a contraction T s.t.
$aT = T\sigma(a)$
all $a \in C(X)$.

Theorem (Peters 84) $\phi : X \to X$ surjective
can exten $\tilde{\phi} : \tilde{X} \to \tilde{X}$ homeomorphism
and
$$C^*_{env}(C(X) \times_\sigma \mathbb{Z}^+) = C(\tilde{X}) \times_{\tilde{\phi}} \mathbb{Z}$$

Theorem (Kakariadis 2016) $\phi : X \to X$ minimal
$\Leftrightarrow$ $C^*_{env}(C(X) \times_\sigma \mathbb{Z}^+)$ simple