Groupoid C*-algebras

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The C*-game

Input $\rightarrow$ C^*-algebra $\rightarrow$ K-theory

Input	C^*-algebra	K-theory
A compact Hausdorff space, X	$C(X)$	$K_*(C(X))$
A group, G	$C^*(G)$	$K_*(C^*(G))$
An action of G on X	$C(X) \rtimes G$	$K_*(C(X) \rtimes G)$

Remarks:

1. Going from a C^*-algebra to its K-theory is not a technical issue but it almost certainly a computational issue (see Mark Tomforde's talk).
2. Going from an input to a C^*-algebra is a technical issue (e.g., given an input how do we construct a C^*-algebra, in the second line should we take the reduced or full group C^*-algebra?).

Input to C^*-algebra

Big Picture: Our approach will be based on the fact that many classes of inputs naturally lead to groupoids and will discuss a method for constructing C^*-algebras from (certain) groupoids.

More concrete plan: We will play the C^*-game when the input is an equivalence relation.

Equivalence relations

Let $\sim$ denote an equivalence relation on a nonempty set X, so that

1. for each $x \in X$, $x \sim x$;
2. if $x \sim y$, then $y \sim x$;
3. if $x \sim y$ and $y \sim z$, then $x \sim z$.

The equivalence class of x is denote by $[x]$ and is the set

$$\{y \in X \mid x \sim y\}.$$

We view an equivalence relation as a subset of $X \times X$ via

$$R = \{(x, y) \in X \times X \mid x \sim y\}$$

$$X/ \sim = \{ [x] \mid x \in X\}$$

Examples when X is a finite set

Consider the following equivalence relations:

Let $X_1 = \{x_1, x_2\}$ and $R_1 = \{(x_1, x_1), (x_1, x_2), (x_2, x_1), (x_2, x_2)\}$.

Let $X_2 = \{x_1, x_2, x_3, x_4\}$ and R_2 be the equivalence relation with all elements equivalent to each other.

Let $X_3 = \{x_1, x_2, x_3, x_4\}$ and R_3 be the equivalence relation generated by $x_1 \sim x_2$ and $x_3 \sim x_4$.

Let $X_4 = \{x_1, x_2\}$ and $R_4 = \{(x_1, x_1), (x_2, x_2)\}$.

Making a C^*-algebra from R: First Attempt

First attempt at making a C^*-algebra:

For each R_i we have that $X_i/\sim_i$ is a compact, Hausdorff space so we can consider $C(X_i/\sim_i)$.

Why is this not a great choice?

It forgets a lot of information about the input. For example,

$$C(X_1/\sim_1) = C(X_2/\sim_2) \cong \mathbb{C}$$

Likewise $C(X_3/\sim_3)$ and $C(X_4/\sim_4)$ are the same.

Making a C^*-algebra from R: Second Attempt

Given a finite set X and an equivalence relation $R \subset X \times X$, we let

$$C^*(R) = \{f : R \to \mathbb{C}\}$$

with

$$(\lambda \cdot f)(x, y) = \lambda f(x, y)$$
$$(f + g)(x, y) = f(x, y) + g(x, y)$$

The star operation when X is finite

Define $(f^*)(x,y) = \overline{f(y,x)}$.

Exercise: Prove that the star operation is well-defined.

Exercise: Would it be well-defined if R was an arbitrary subset of $X \times X$ rather than an equivalence relation?

Multiplication in $C^*(R)$ when X is finite

The convolution product on $C^*(R)$ is defined via

$$(f * g)(x,y) = \sum_{z \sim x} f(x,z)g(z,y)$$

Exercise: Prove that $*$ is well-defined. Would it be well-defined if R was an arbitrary subset of $X \times X$ rather than an equivalence relation?

Exercise: Prove $C^*(R)$ is a $*$-algebra.

Somewhat involved exercise: Define a norm on $C^*(R)$ (still with X finite) so that it becomes a C^*-algebra. (Hint: the examples on the next slide should be helpful).

Examples

Back to R_1, which was defined by taking $X_1 = \{x_1, x_2\}$ and

$$R_1 = \{(x_1, x_1), (x_1, x_2), (x_2, x_1), (x_2, x_2)\}$$

and

$$C^*(R_1) = \{f : R_1 \to \mathbb{C}\}$$

As a vector space $C^*(R_1)$ is four dimensional; it has basis:

$$f_{ij}(x_l, x_m) = \begin{cases} 1 & l = i \text{ and } m = j \\ 0 & \text{otherwise} \end{cases}$$

where $1 \leq i \leq 2$ and $1 \leq j \leq 2$.

Examples

Exercise: Prove that $C^*(R_1) \cong M_2(\mathbb{C})$ as $*$-algebras via the map defined on the above basis by

$$f_{11} \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, f_{12} \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, f_{21} \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, f_{22} \mapsto \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Exercise: Proved that $C^*(R_2) \cong M_4(\mathbb{C})$, $C^*(R_3) \cong M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$ and $C^*(R_4) \cong \mathbb{C}^2$.

The C*-game in this case

Definition

A C^*-algebra A is finite dimensional if it is finite dimensional as a vector space.

Theorem

If X a finite nonempty set and $R \subseteq X \times X$ is an equivalence relation, then $C^*(R)$ is finite dimensional.

Theorem

If A is a C^*-algebra that is finite dimensional, then there exists X finite and an equivalence relation $R \subseteq X \times X$ such that $A \cong C^*(R)$.

Exercise: (Still with X finite) compute the K-theory of $C^*(R)$ in terms of the equivalence relation R.

Generalizing to the case when X is infinite

To generalize to the case when X is infinite, we need more structure on R.

One can work more generally but we will assume R is a locally compact, Hausdorff space.

Let $r : R \to X$ be defined via $(x, y) \mapsto x$ and $s : R \to X$ be defined via $(x, y) \mapsto y$.

Definition

A topological equivalence relation R is an equivalence relation over X with a locally compact Hausdorff topology such that

$$(x, y) \mapsto (y, x) \text{ and } ((x, y), (y, z)) \mapsto (x, z)$$

are continuous. This implies that r and s are both continuous (see the next slide).

Important Remark

The topology on R determines the topology on X (rather than the opposite). How?

Using the identification of sets: $X \cong \{(x, x) \mid x \in X\} \subseteq R$.

Based on this, r is more correctly defined as $r : R \to R$ via

$$(x, y) \mapsto (x, x)$$

Exercise: Based on the definition of r, define s.

Exercise: Prove that r and s are continuous.

Étale equivalence relations

Definition

A topological equivalence relation R is étale if r and s are local homeomorphisms.

Theorem

If R is étale, then for each $x \in X$

$$\{(x, y) \mid y \sim x\} \text{ and } \{(y, x) \mid y \sim x\}$$

are discrete subsets of R.

Examples

1. If X is nonempty and finite, then any equivalence relation R on X is étale.
2. Suppose $\pi : X \to Y$ is a local homeomorphism and
$$R_\pi := \{(x_1, x_2) \in X \times X \mid \pi(x_1) = \pi(x_2)\}.$$
Then $R_\pi \subseteq X \times X$ is étale.
3. As specific case of the previous example, take $\pi : \mathbb{R} \to S^1$ the standard covering map.

Exercise: What is missing from the above examples to make the statements "is étale" precise?

An example from dynamical systems

Definition

A dynamical system is a pair (X, φ) where X is a compact Hausdorff space and $\varphi : X \to X$ is a homeomorphism.

We say that (X, φ) is free if the following property holds: $\varphi^n(x) = x$ for some $x \in X$ if and only if $n = 0$.

Definition

Suppose (X, φ) is free. Define

$$R_{orbit} = \{(x_1, x_2) \in X \times X \mid \varphi^n(x_1) = x_2 \text{ for some } n \in \mathbb{Z}\}$$

Exercise: Prove R_{orbit} is an equivalence relation.

Étale topology on R_{orbit}

Using the assumption that (X, φ) is free, we can (as sets) identify R_{orbit} with $X \times \mathbb{Z}$ via

$$(x, n) \in X \times \mathbb{Z} \mapsto (\varphi^n(x), x) \in R_{orbit}.$$

Since $X \times \mathbb{Z}$ has a natural topology this gives R_{orbit} a topology.

Facts:

1. The original topology on X agrees with the topology on $X \times \{0\} \subseteq X \times \mathbb{Z}$.
2. X is compact but R_{orbit} is not compact.
3. The topology on R_{orbit} is not the subspace topology from $R_{orbit} \subseteq X \times X$.
4. Using the topology above, R_{orbit} is étale (but with the subspace topology is not).

Example of an equivalence relation that is not étale

Example

Let $X = [0, 1]$ and define an equivalence relation

$$R = \{(x, x) \mid x \in X\} \cup \{(0, 1), (1, 0)\}.$$

where we give $R \subseteq X \times X$ the subspace topology.

Exercise: Prove that for each $x \in X$

$$\{(x, y) \mid (x, y) \in R\} \text{ and } \{(y, x) \mid (y, x) \in R\}$$

are finite (and hence discrete) subsets of R.

Exercise: Prove that R is not étale.

Functions of compact support

Suppose R is a locally compact Hausdorff étale equivalence relation. Let

$$C_c(R) = \{f : R \to \mathbb{C} \mid f \text{ is continuous and has compact support}\}$$

We have the following algebraic operations:

$$\begin{aligned}
(\lambda \cdot f)(x, y) &= \lambda f(x, y) \\
(f + g)(x, y) &= f(x, y) + g(x, y) \\
(f^*)(x, y) &= \overline{f(y, x)} \\
(f * g)(x, y) &= \sum_{z \sim x} f(x, z) g(z, y)
\end{aligned}$$

Very involved exercise: Prove that the convolution product is well-defined.
Exercise: Prove that the convolution product is **not** well-defined for the non-étale equivalence relation from the previous slide.

C^*-algebras associated to an étale equivalence relation

Plan: Complete $C_c(R)$ to get a C^*-algebra.

The issue is what norm should we complete with respect to?

There is no canonical choice.

In general, given a faithful representation $\phi : C_c(R) \to \mathcal{B}(\mathcal{H})$, we can form

$$C^*_\phi(R) := \overline{C_c(R)}^{||\cdot||}$$

where $||\cdot||$ is the operator norm in $\mathcal{B}(\mathcal{H})$.

We will discuss representations in more detail later in the talk.

If (X, φ) is free, then $C^*(R_{orbit}) \cong C(X) \rtimes \mathbb{Z}$.

An explicit example

Take $X = \{0,1\}^{\mathbb{N}}$ with the product topology (X is the Cantor set) and φ the odometer homeomorphism (i.e., φ acts via add one to the first coordinate and then carry over). That is,

$$\varphi(0, x_1, x_2, \ldots) = (1, x_1, x_2, \ldots)$$
$$\varphi(1, 0, x_2, \ldots) = (0, 1, x_2, \ldots)$$
$$\varphi(1, 1, 0, x_3, \ldots) = (0, 0, 1, x_3, \ldots)$$

and

$$\varphi(1, 1, 1, \ldots) = (0, 0, 0, \ldots)$$

Goal: Understand the orbit relation and the C^*-algebra associated to the orbit relation for this dynamical system.

Example of an orbit

The orbit of $x \in X$ is the set

$$\{y \in X \mid y = \varphi^n(x) \text{ for some } n \in \mathbb{Z}\}$$

By the definition of the orbit relation, $[x]_{orbit}$ is exactly the orbit of x.

For example, when $x = (0, 1, 0, 1, 0, \ldots)$, we have

$$\varphi(x) = (1, 1, 0, 1, 0, \ldots)$$

$$\varphi^2(x) = (0, 0, 1, 1, 0, \ldots)$$

$$\varphi^{-1}(x) = (1, 0, 0, 1, 0, \ldots)$$

Another relation on X

Recall that $X = \{0,1\}^{\mathbb{N}}$ with the product topology.

Definition

We say that $x = (x_0, x_1, x_2, \ldots)$ and $y = (y_0, y_1, y_2, \ldots)$ are tail equivalent and write

$$x \sim_{tail} y$$

if there exists $N \in \mathbb{N}$ such that $x_i = y_i$ for all $i \geq N$.

Q: Is orbit equivalence relation from the odometer action the same as the tail equivalence relation?

A: Not quite

$(1,1,1,\ldots) \sim_{orbit} (0,0,0,\ldots)$ (since $\varphi(1,1,1,\ldots) = (0,0,0,\ldots)$) but $(1,1,1,\ldots) \not\sim_{tail} (0,0,0,\ldots)$

R_{tail} vs R_{orbit}

The relationship between these two equivalence relations is the following:

1. If $x \in X$ and $x \not\sim_{orbit} (0,0,0,\ldots)$, then

$$[x]_{tail} = [x]_{orbit}$$

2. The orbit of $(0,0,0,\ldots)$ has been broken into its forward and backward parts:

$$[(1,1,1,\ldots)]_{tail} \cup [(0,0,0,\ldots)]_{tail} = [(0,0,0,\ldots)]_{orbit}$$

3. $R_{tail} \subseteq R_{orbit}$ as an open subrelation.

Exercise: Prove that if $\hat{R}$ is an open subrelation of R and R is étale, then $\hat{R}$ (with the subspace topology) is also étale.

$C^*(R_{tail})$ vs $C^*(R_{orbit})$

Define $\iota : C_c(R_{tail}) \to C_c(R_{orbit})$ by

$$\iota(f)(x_1, x_2) = \begin{cases} f(x_1, x_2) & (x_1, x_2) \in R_{tail} \\ 0 & (x_1, x_2) \notin R_{tail} \end{cases}$$

Fact: ι can be extended to a $*$-homomorphism $C^*(R_{tail}) \to C^*(R_{orbit})$.

Exercise: Prove that $C^*(R_{tail}) \to C^*(R_{orbit})$ is injective (so that we can view $C^*(R_{tail})$ as a subalgebra of $C^*(R_{orbit})$).

Fact: $C^*(R_{tail})$ is a large subalgebra of $C^*(R_{orbit}) \cong C(X) \rtimes \mathbb{Z}$. (see Dawn Archey's talk)

The structure of $C^*(R_{tail})$

$C^*(R_{tail})$ is the CAR algebra!

Outline of the ideas of the proof:

Define $\mathbb{C} \to C^*(R_{tail})$ via $\lambda \mapsto \lambda I$ where $I \in C_c(R_{tail})$ is defined by

$$I(x, \hat{x}) = \begin{cases} 1 & x = \hat{x} \\ 0 & \text{otherwise} \end{cases}$$

Exercise: Prove that $\mathbb{C} \to C^*(R_{tail})$ is a well-defined injective $*$-homomorphism.

Next we want to define $M_2(\mathbb{C}) \to C^*(R_{tail})$.

The map: $M_2(\mathbb{C}) \to C^*(R_{tail})$

For i, $j \in \{0, 1\}$ define

$$E_{ij} = \{(x, \hat{x}) \mid \pi_0(x) = i, \pi_0(\hat{x}) = j \text{ and for } k > 0, \pi_k(x) = \pi_k(\hat{x})\}$$

where $\pi_k : \{0, 1\}^{\mathbb{N}} \to \{0, 1\}$ is the projection onto the k-coordinate.

For example

$$\begin{aligned} ((0, 1, 1, \ldots), (1, 1, 1, \ldots)) &\in E_{01} \\ ((0, 1, 1, \ldots), (0, 1, 1, \ldots)) &\notin E_{01} \\ ((0, 1, 1, \ldots), (1, 0, 0, \ldots)) &\notin E_{01} \\ ((0, 1, 0, \ldots), (1, 0, 0, \ldots)) &\in E_{01} \\ ((0, 1, 0, 1, 0, \ldots), (1, 1, 0, 1, 0, \ldots)) &\in E_{01} \end{aligned}$$

Note: One of these ordered pairs is not even in R_{tail}.

The map: $M_2(\mathbb{C}) \to C^*(R_{tail})$

$E_{ij} = \{(x, \hat{x}) \mid \pi_0(x) = i, \pi_0(\hat{x}) = j$ and for $k > 0, \pi_k(x) = \pi_k(\hat{x})\}$ where $\pi_k : \{0,1\}^{\mathbb{N}} \to \{0,1\}$ is the projection onto the k-coordinate.

Define $M_2(\mathbb{C}) \to C^*(R_{tail})$ via

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto a\chi_{00} + b\chi_{01} + c\chi_{10} + d\chi_{11}$$

where $\chi_{ij} : R_{tail} \to \mathbb{C}$ is defined via

$$\chi_{ij}(x, \hat{x}) = \begin{cases} 1 & (x, \hat{x}) \in E_{ij} \\ 0 & \text{otherwise} \end{cases}$$

Exercise: Prove that $M_2(\mathbb{C}) \to C^*(R_{tail})$ is a well-defined injective $*$-homomorphism.

The structure of $C^*(R_{tail})$

Exercise: How are the maps $\mathbb{C} \to C^*(R_{tail})$ and $M_2(\mathbb{C}) \to C^*(R_{tail})$ related? (Hint: What is $\chi_{00} + \chi_{11}$?)

Involved exercise: For each $n \in \mathbb{N}$, define the relevant injective $*$-homomorphism $M_{2^n}(\mathbb{C}) \to C^*(R_{tail})$.

Very involved exercise: Prove that $\cup_{n\in\mathbb{N}} M_{2^n}(\mathbb{C})$ is dense in $C^*(R_{tail})$.

Exercise: Is $\cup_{n\in\mathbb{N}} M_{2^n}(\mathbb{C}) = C_c(R_{tail})$?

Summary: Using the fact that $C^*(R_{tail})$ is large inside $C^*(R_{orbit})$ reduces many questions about $C^*(R_{orbit})$ to $C^*(R_{tail})$ and we know "everything" about $C^*(R_{tail})$ because it is the CAR algebra.

What is a groupoid?

Definition

A groupoid is a nonempty set $\mathcal{G}$ with the following additional structure:

1. a fixed subset of $\mathcal{G} \times \mathcal{G}$ denoted by $\mathcal{G}^2$;
2. a map $\mathcal{G}^2 \to \mathcal{G}$ denoted by $(g, h) \mapsto gh$;
3. an involution $\mathcal{G} \to \mathcal{G}$ denoted by $g \mapsto g^{-1}$.

such that

1. if g, h and k are in $\mathcal{G}$ with (g, h), (h, k) both in $\mathcal{G}^2$, then (gh, k). (g, hk) are both in $\mathcal{G}^2$ and $(gh)k = g(hk)$ (so that we can simply write ghk);
2. for each $g \in \mathcal{G}$, both (g, g^{-1}) and (g^{-1}, g) are in $\mathcal{G}^2$ and moreover if $(g, h) \in \mathcal{G}^2$, then $g^{-1}gh = h$ and if $(h, g) \in \mathcal{G}^2$, then $hgg^{-1} = h$.

An equivalence relation is a groupoid

Let X be a nonempty set and $R \subseteq X \times X$ be an equivalence relation. We take

1. $\mathcal{G} = R$;
2. $\mathcal{G}^2 = \{((x, y), (a, z)) \in R \times R \mid y = a\}$;
3. $\mathcal{G}^2 \to \mathcal{G}$ defined via $((x, y), (y, z)) \mapsto (x, z)$;
4. $\mathcal{G} \to \mathcal{G}$ defined via $(x, y) \mapsto (y, x)$.

Exercise: Prove that the above defines a groupoid.

More structure

Definition

Let $\mathcal{G}$ be a groupoid. The set of units of $\mathcal{G}$ is

$$\mathcal{G}^0 = \{g^{-1}g \mid g \in G\}$$

Define the range map, $r : \mathcal{G} \to \mathcal{G}^{(0)}$ via $g \mapsto gg^{-1}$ and the source map, $s : \mathcal{G} \to \mathcal{G}^{(0)}$ via $g \mapsto g^{-1}g$.

For the groupoid associated to an equivalence relation, we have

$$\mathcal{G}^0 = \{(y,x)(x,y) \mid (x,y) \in \mathcal{G}\} = \{(y,y) \mid y \in X\} \cong X$$

Moreover,

$$r(x,y) = (x,y)(y,x) = (x,x)$$

and

$$s(x,y) = (y,x)(x,y) = (y,y)$$

More examples

Example

Given a nonempty set X, we can define a groupoid by taking

1. $\mathcal{G} = X$,
2. $\mathcal{G}^2 = \{((x_1, x_2) \in X \times X \mid x_1 = x_2\}$,
3. $\mathcal{G}^2 \to \mathcal{G}$ defined via $(x, x) \mapsto x$ and
4. $\mathcal{G} \to \mathcal{G}$ defined via $x \mapsto x$.

Exercise: How are the constructions of a groupoid from a set and an equivalence relation related? (Hint: One is a special case of the other).

Exercise: What is $\mathcal{G}^0$ in this case?

More examples

Example

Given a group G, we can define a groupoid by taking

1. $\mathcal{G} = G$,
2. $\mathcal{G}^2 = G \times G$,
3. $\mathcal{G}^2 \to \mathcal{G}$ defined by group multiplication and
4. $\mathcal{G} \to \mathcal{G}$ defined by taking the inverse.

Exercise: What is $\mathcal{G}^0$ in this case?

Topological groupoids

Definition

A topological groupoid is a groupoid with a (locally compact, Hausdorff) topology on $\mathcal{G}$ such that

1. $\mathcal{G}^2$ is given the subspace topology from $\mathcal{G} \times \mathcal{G}$;
2. $\mathcal{G}^2$ is closed;
3. $\mathcal{G}^2 \to \mathcal{G}$ is continuous;
4. $\mathcal{G} \to \mathcal{G}$ is continuous.

Examples? Same as above but with topologies, so topological spaces (rather than sets), topological groups (note: discrete groups are topological groups), and topological equivalence relations.

Exercise: Prove that given a topological groupoid the range and source maps are continuous.

Étale groupoids

Definition

A topological groupoid is étale if r and s are local homeomorphisms.

Example

The groupoid associated to a topological group, G, is étale if and only if G is discrete.

Example

If X is a locally compact and Hausdorff space, then the associated groupoid is étale.

Functions of compact support

Suppose that $\mathcal{G}$ is a locally compact, Hausdorff, étale groupoid. Let

$$C_c(\mathcal{G}) = \{a : \mathcal{G} \to \mathbb{C} \mid a \text{ is continuous and has compact support}\}.$$

Algebraic operations:

$$\begin{aligned}(\lambda \cdot a)(g) &= \lambda a(g) \\ (a + b)(g) &= a(g) + b(g) \\ (a^*)(g) &= \overline{a(g^{-1})} \\ (a * b)(g) &= \sum_{r(h)=r(g)} a(h)b(h^{-1}g)\end{aligned}$$

Defining representations

Theorem

Let u be a unit in a locally compact, Hausdorff étale groupoid $\mathcal{G}$ (i.e., $u \in \mathcal{G}^0$). For each $a \in C_c(G)$ and $\xi \in \ell^2(s^{-1}(u))$ the expression

$$(\pi_\lambda^u(a)\xi)(g) = \sum_{r(h)=r(g)} a(h)\xi(h^{-1}g)$$

defines an element in $\ell^2(s^{-1}(u))$. Moreover, $\pi_\lambda^u(a)$ is in $\mathcal{B}(\ell^2(s^{-1}(u)))$ with $||\pi_\lambda^u(a)||$ bounded by a constant that is independent of u (it is does depend on a), and $\pi_\lambda^u : C_c(\mathcal{G} \to \mathcal{B}(\ell^2(s^{-1}(u)))$ is a $*$-representation.

The C^*-algebra associated to a groupoid

Theorem

Suppose that $\mathcal{G}$ is a locally compact, Hausdorff, étale groupoid. Then

$$||a|| := \sup\{||\pi(a)|| \mid \pi \text{ a representation of } C_c(\mathcal{G})\}$$

and

$$||a||_\lambda := \sup\{||\pi_\lambda^u(a)|| \mid u \in \mathcal{G}^0\}$$

define (non-complete) C^*-norms on $C_c(\mathcal{G})$.

Definition

The full groupoid C*-algebra of $\mathcal{G}$ is $C^*(\mathcal{G}) = \overline{C_c(\mathcal{G})}^{||\cdot||}$.

Definition

The reduced C*-algebra of $\mathcal{G}$ is $C^*_\lambda(\mathcal{G}) = \overline{C_c(\mathcal{G})}^{||\cdot||_\lambda}$.

What's next?

For more, see

1. Ian Putnam's lecture notes on C^*-algebra (available on his website, see Chapter 3);
2. Aidan Sims "Étale groupoids and their C^*-algebras" arXiv:1710.10897;
3. Karen Strung "An introduction to C^*-algebras and the Classification Programme" (available on her website, see Exercises 9.6.12 and 9.6.13 for more on the odometer action and its C^*-algebra);
4. J.N. Renault "A groupoid approach to C^*-algebras".

Thank you!