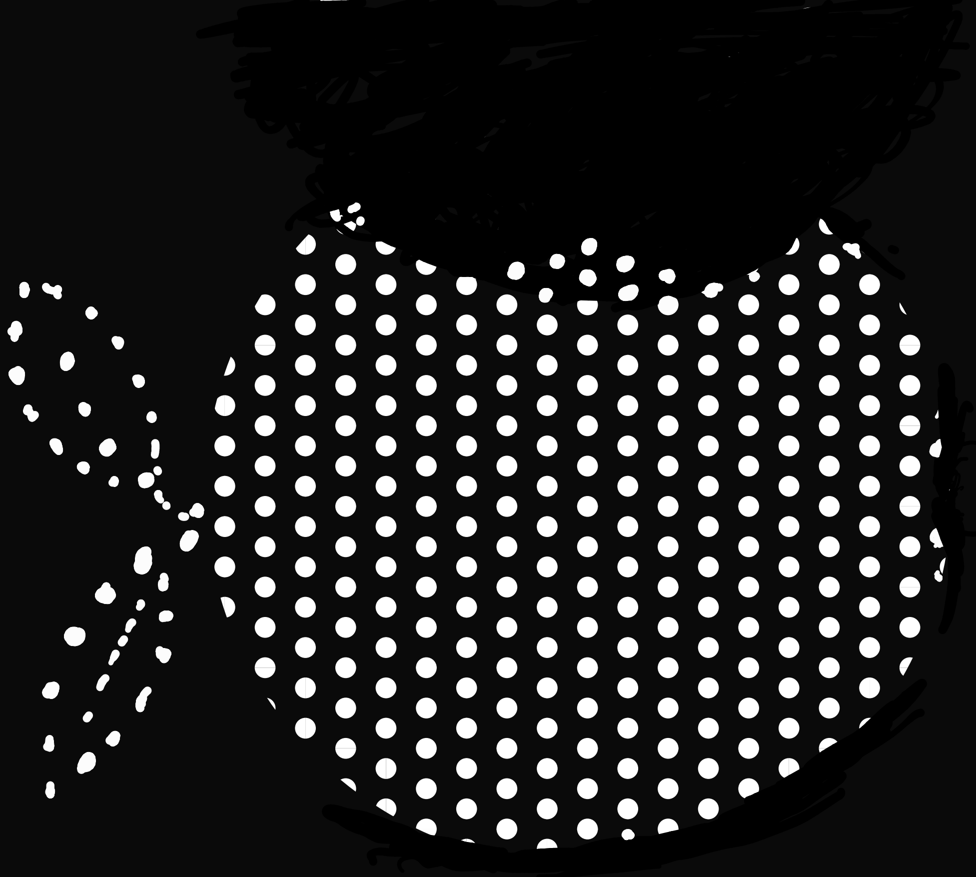

GOALS

RIGIDITY IN GROUP VON NEUMANN ALGEBRA

(joint with ALEC DIAZ-ARIAS and DANIEL DRIMBE)

G – countable discrete group

$\ell^2 G = \{\xi : G \to \mathbb{C} \,/\, \sum_{g\in G} |\xi(g)|^2 < \infty\}$ ← Hilbert space.

⇝ left regular rep. $u : G \to \mathcal{U}(\ell^2 G)$ given by

$$u_g(\xi)(h) = \xi(g^{-1}h), \quad g,h \in G, \ \xi \in \ell^2 G$$

A) the group von Neumann algebra of G is

$$\mathcal{L}(G) = \{\sum_{\text{finite}} c_g u_g \,/\, c_g \in \mathbb{C}, g \in G\} = \overline{\mathbb{C}[G]}^{SOT} \subset B(\ell^2 G)$$

where

$T_i \xrightarrow{SOT} T$ iff $\|T_i\xi - T\xi\| \longrightarrow 0 \quad \forall \xi \in \ell^2 G$

⇝ $\tau : \mathcal{L}(G) \to \mathbb{C}$ normal state $(\tau(x) = \langle x\delta_e, \delta_e\rangle)$

- faithful $\tau(x^*x) = 0 \iff x = 0$
- tracial $\tau(xy) = \tau(yx) \quad \forall x,y \in \mathcal{L}(G)$

⇝ $\mathcal{L}(G)$ is a finite von Neumann algebra $(vv^* = 1 \iff v^*v = 1)$

<u>THM</u> (Murray–von Neumann '43)

$\mathcal{L}(G)$ is a II_1 factor $(\mathcal{Z}(\mathcal{L}(G)) = 1)$ iff

$\forall g \in G \setminus \{e\}$ the conjugacy class $g^G = \{hgh^{-1} / h \in G\}$ is infinite

i.e. G is icc.

Examples a) $\mathbb{F}_n$, $n \geq 2$; $\Gamma_1 * \Gamma_2$, $|\Gamma_1| \geq 2$, $|\Gamma_2| \geq 3$.

b) lamplighter group, $G_\infty = \bigcup_{n\in\mathbb{N}} G_n$

c) wreath products $A \wr_{H/B} H$, $[H:B] = \infty$

d) $PSL_n(\mathbb{Z})$, $n \geq 2$

e) uniform lattices $\Gamma < Sp(n,1)$, $n \geq 2$ where

$Sp(n,1) = \{A \in M_{n+1}(\mathbb{H}) \,/\, A^*JA = J\}$ where $J = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}$

B) the reduced C^*-algebra of G is

$$C_r^*(G) = \overline{\mathbb{C}[G]}^{\|\cdot\|_\infty} \subset B(\ell^2 G) \text{ where}$$

$T_i \xrightarrow{\|\cdot\|_\infty} T$ iff $\sup_{\|\xi\| \leq 1} \|T_i\xi - T\xi\| \longrightarrow 0$

⇝ $\mathbb{C}[G] \subset C_r^*(G) \subset \mathcal{L}(G)$

MAIN THEME OF STUDY

⇝ How much information does $\mathcal{L}(G)$ remembers of G?

⇝ Is it possible to identify a comprehensive list of canonical properties of G that are completely recognizable from $\mathcal{L}(G)$?

⇝ Can G be completely remembered by $\mathcal{L}(G)$?

SOME NON-RESULTS:

1. (folk) If G and H infinite abelian then
$$L(G) \cong L(H) \cong L^\infty[0,1]$$
2. (Connes '76) If G and H are amenable icc. then $L(G) \cong L(H) \cong \overline{\bigcup_n M_{2^n}(\mathbb{C})}^{SOT} = \mathcal{R}$

Concrete examples:
$$L(\mathbb{Z} \wr \mathbb{Z}) \cong L(\mathbb{Z}_2 \wr \mathbb{Z}) \cong L(\mathfrak{S}_\infty)$$
3. (Dykema '93) If G_i and H_i are infinite amenable then
$$L(\underbrace{G_1 * G_2 * \dots * G_n}_{2 \times \mathbb{Z}}) \cong L(H_1 * H_2 * \dots * \underbrace{H_n}_{\mathbb{Z}})$$

CONCLUSION: In general, no memory of classical group invariants: torsion, rank, gen. & rel.

SOME RESULTS:

1. (Murray-von Neumann '43) $L(\mathbb{F}_2) \not\cong L(\mathbb{F}_2 \times \mathfrak{S}_\infty)$
2. (McDuff '69) $\exists$ a continuum of non-isomorphic group factors.
3. (Cowling-Haagerup '89) $G < Sp(n,1)$, $H < Sp(m,1)$ unif. lattices with $n \neq m \Rightarrow L(G) \not\cong L(H)$.
4. $L(\mathbb{F}_n) \not\cong L(A \rtimes \Gamma)$ (Voiculescu '96) $\forall A$ abelian infinite
$\not\cong L(\Gamma_1 \times \Gamma_2)$ (Ge '98) $\forall \Gamma_i$ infinite
5. Second assertion holds if $\mathbb{F}_n$ is replaced by any non-elem. hyperbolic group (Ozawa 03)

USING POPA DEFORMATION/RIGIDITY THEORY

6. $L(G_1 * G_2 * \dots * G_n) \cong L(H_1 * H_2 * \dots * H_m) \Rightarrow$
$\Rightarrow n = m$ and $\exists \sigma \in \mathfrak{S}_n$ s.t. $L(G_i) \cong L(H_{\sigma(i)})$

a) G_i, H_j are prop(T) groups (Ioana-Peterson-Popa '05)

b) G_i, H_j non-amenable direct products (C-Houdayer '08)

c) G_i, H_j admit infinite normal amenable subgroups (Ioana '12)

7. if $L(\mathbb{F}_n) \cong L(H)$ then all infinite amenable subgroup $B \leq H$ we have that normalizer $N_H(B)$ is amenable as well (Ozawa-Popa 07)

MAJOR OPEN PROBLEMS

(Murray-von Neumann '43)

$$\mathcal{L}(\mathbb{F}_n) \cong \mathcal{L}(\mathbb{F}_m) \Longrightarrow n=m \,?$$

(Connes '80s)

$$\mathcal{L}(PSL_n(\mathbb{Z})) \cong \mathcal{L}(PSL_m(\mathbb{Z})) \Longrightarrow n=m\,?$$

$$n,m \geq 3$$

W^*-SUPERRIGIDITY AND C^*-SUPERRIGIDITY

1. A group G is called W^*-superrigid if it satisfies the following rigidity statement: whenever H is an arbitrary group and $\theta : \mathcal{L}(G) \to \mathcal{L}(H)$ is an arbitrary $*$-isomorphism then $\exists$: $\delta : G \to H$ group isom, $\eta : G \to \mathbb{T}$ multiplicative character, $\to \{ |z|=1 \mid z \in \mathbb{C} \}$ $w \in \mathcal{L}(H)$ unitary $\quad u_g \to \eta(g) v_{\delta(g)}$, such that

$$\theta(u_g) = \eta(g)\, w\, v_{\delta(g)}\, w^* \quad \forall g \in G.$$

Here $\{u_g \mid g \in G\}'' = \mathcal{L}(G)$ and $\{v_h \mid h \in H\}'' = \mathcal{L}(H)$ are the canonical unitaries.

- In this situation $\mathcal{L}(G)$ completely remembers G!!!

2. Similarly, G is C^*-superrigid if whenever H is an arbitrary group and $\theta : C^*_r(G) \to C^*_r(H)$ is an arbitrary $*$-isomorphism then there exist

a) $\delta : G \to H$ group isom,

b) $\eta : G \to \mathbb{T}$ multiplicative character,

c) $w \in \mathcal{L}(H)$ $C^*_r(H)$? unitary

such that $\theta(u_g) = \eta(g)\, w\, v_{\delta(g)}\, w^* \quad \forall g \in G$.

$\rightsquigarrow$ Condition c) above is optimal. (Phillips '87) proved that there are uncountably many unitaries $w \in \mathcal{L}(H)$ that implement outer automorphisms of $C^*_r(H)$.

CONJECTURE (Connes '80s, Popa '07)

Any icc property (T) group is W^*-superrigid.

$\rightsquigarrow$ Major open problem in the field; no such examples are known to this day.

Examples of W^* and C^*-superrigid groups

(Ioana-Popa-Vaes '10) generalized wreath products

$B < K$ ← icc, biexact, property (T) groups

↑ infinite amenable malnormal group

$$|B \cap gBg^{-1}| < \infty \quad \forall\, g \in K \setminus B$$

consider the generalized Bernoulli action

$K \curvearrowright^{\sigma} \mathbb{Z}_2^{(K/B)} = \bigoplus_{K/B} \mathbb{Z}_2$ given by

$$\sigma_g\big((x_i)_{i\in K/B}\big) = (x_{g^{-1}i})_{i \in K/B}.$$

Consider the semidirect product

$$G = \mathbb{Z}_2 \wr_B K \qquad \mathbb{Z}_3 \wr_B K$$

THM (IPV10): G is W^*-superrigid.

⇝ landmark result; introduction of analysis of comultiplication; height techniques.

(C-Ioana '16) amalgamated free products

⇝ $B \leq K$ ← biexact group

↑ icc, amenable

$$QN_K^{(1)}(B) := \{g \in K \mid [B : gBg^{-1} \cap B] < \infty\} = B$$

$\forall g \in K$ and $\forall\, A \leq_f gBg^{-1} \cap B$ the centralizer $C_K(A) = 1$

$$G = (K \times K) *_{\Delta(B)} (K \times K) \qquad \boxed{K = \mathbb{Z} \wr \mathbb{F}_n,\ B = \mathbb{Z} \wr \mathbb{Z}}$$

$\Delta(B) = \{(b,b) \mid b \in B\} < K \times K$ ← diag. group

THM (CI '16) G is W^* and C^*-superrigid

⇝ first example of C^*-superrigid group; other examples of C^*-reconstructible groups: Bieberbach groups (Knudby-Raum-Thiel-White '16), 2-step nilpotent groups (Eckhart-Raum '18), free nilpotent (Omland '19);

PROP If G is W^*-superrigid and $C^*_r(G)$ has the unique trace property then G is C^*-superrigid.

⇝ by (Breuillard-Kalantar-Kennedy-Ozawa '14) it suffices to check that G has trivial amenable radical

NEW RESULTS (C-(Diaz-Arias)-Drimbe '20)

$\boxed{\text{Class } \mathcal{S}:}$ semidirect products with nonamenable core

K - icc, torsion free, biexact property (T) group

Examples:
- torsion free, hyperbolic, prop (T) group (e.g. $\forall$ uniform lattice $K < Sp(n,1)$ $n \geq 2$)
- torsion free, prop (T) group that is hyperbolic rel to any family of amenable subgroups. (constructions in geometric group theory by (Arzhantseva-Minasyan-Osin '06)

let $K_1, K_2, K_3, \dots, K_n$ copies of K, $n \geq 2$

$K \overset{\rho^i}{\curvearrowright} K_i$ by conjugation $\rho^i_g(h) = ghg^{-1}$, $g \in K$, $h \in K_i$

$K \overset{\rho}{\curvearrowright} K_1 * K_2 * \dots * K_n$ by free product automorphism

$\rho_g = \rho^1_g * \rho^2_g * \dots * \rho^n_g$

$$\boxed{\rho_g(abab) = \rho^1_g(a)\rho^2_g(b)\rho^1_g(a)\rho^2_g(b)}$$

consider the semidirect product

$$G = (K_1 * K_2 * \dots * K_n) \rtimes_\rho K \in S$$

THM: Any group $G \in S$ is W^* and C^*-superrigid

⇝ $|S| = \aleph_1$; first residually finite examples

$\boxed{\text{Class } \mathcal{T}_0}$ direct product groups

$$JPV = \{ \mathbb{Z}_2 \wr_B K \mid K \text{ icc, biexact, prop}(T),\ B < K \text{ malnormal} \}$$

$$\mathcal{T}_0 = \{ G_1 \times G_2 \times \dots \times G_n \mid G_i \in JPV,\ n \geq 2 \}$$

THM: Any group $G \in \mathcal{T}_0$ is W^*-superrigid

⇝ first established a product rigidity result for biexact groups similar (C-deSantiago-Sinclair '15)

$\mathcal{L}(G_1 \times G_2 \times \dots \times G_n) \cong \mathcal{L}(H) \Rightarrow H = H_1 \times H_2 \times \dots \times H_n$ and up amplifications $\mathcal{L}(G_i) \cong \mathcal{L}(H_i)$.

⇝ $n=2$ this was already done; new proof.

⇝ use (IPV10) to conclude.

$\boxed{\text{Class } \mathcal{T}}$ - tree groups - constructed iteratively from class $\mathcal{T}_0$ using amalgams and HNN extensions

$\mathcal{T}_1$:

i) $G = G_1 *_A G_2$, $G_i \in \mathcal{T}_0$, $A \leq G_i$ icc amenable
$QN^{(1)}_{G_i}(A) = A$

ii) $G = K*_\varphi = \langle K, t \mid \varphi(a) = tat^{-1} \rangle$, $\varphi : A \to K$ monomorphism
$K \in \mathcal{T}_0$, $A < K$ icc amenable
$QN^{(1)}_K(A) = A$; $[A : A \cap g\varphi(A)g^{-1}] = \infty$, $\forall g \in K$

factors set: $f(G) = \{G_1, G_2\}$ or $\{K\}$

amalgamated subgroups set: $a(G) = \{A\}$

$\vdots$

inductively define:

$\mathcal{T}_{i+1}$:

i) $G = G_1 *_A G_2$, $G_i \in \mathcal{T}_k$, $k \leq i$
$A \leq G_i$ icc amenable; $QN^{(1)}_{G_i}(A) = A$;
$A \in a(G_1) \cup a(G_2)$ or $A \cap B = 1$ $\forall B \in a(G_1) \cup a(G_2)$

ii) $G = K*_\varphi$, $\varphi : A \to K$ monomorphism
$K \in \mathcal{T}_k$, $k \leq i$, $A < K$ icc amenable
$QN^{(1)}_K(A) = A$; $A \in a(K)$ or $A \cap B = 1$ $\forall B \in a(K)$

Case i) $f(G) = f(G_1) \cup f(G_2)$, $a(G) = a(G_1) \cup a(G_2) \cup \{A\}$

ii) $f(G) = f(K)$, $a(G) = a(K) \cup \{A\}$

$$\mathcal{T} = \bigcup_{i \geq 0} \mathcal{T}_i \leftarrow \text{tree groups}$$

Examples: Let $\Gamma \leq Sp(n,1)$ $n \geq 2$ uniform lattice

$B, C, D < \Gamma$ inf. cyclic $B \cap gCg^{-1} = 1$, $B \cap gDg^{-1} = 1, \dots$

$K := (\mathbb{Z}_2 \wr_B \Gamma) \times (\mathbb{Z}_2 \wr_B \Gamma)$ and

$\varphi : (\mathbb{Z}_2 \wr_B C) \times (\mathbb{Z}_2 \wr_B C) \to (\mathbb{Z}_2 \wr_B D) \times (\mathbb{Z}_2 \wr_B D)$ isom

$$\big((\mathbb{Z}_2 \wr_B \Gamma) \times (\mathbb{Z}_2 \wr_B \Gamma)\big)_{*\varphi} \in \mathcal{T}_1$$

$$\Big(\big((\mathbb{Z}_2 \wr_B \Gamma) \times (\mathbb{Z}_2 \wr_B \Gamma)\big)_{*\varphi}\Big) *_{(\mathbb{Z}_2 \wr_B C) \times (\mathbb{Z}_2 \wr_B C)} \big((\mathbb{Z}_2 \wr_B \Gamma) \times (\mathbb{Z}_2 \wr_B \Gamma)\big) \in \widetilde{\mathcal{T}}_2$$

THM Any group $G \in \bigcup_{i \geq 1} \mathcal{T}_i$ is W^* and C^*-superrigid.

COR For any $G \in SU\mathcal{T}$ we have that

$$\frac{Aut(\mathcal{L}(G))}{Inn(\mathcal{L}(G))} = Out(\mathcal{L}(G)) = Char(G) \rtimes Out(G)$$

$w/nn(C_r^*(G)) := \{ \theta \in Aut(C_r^*(G)) \,/\, \exists\, u \in \mathcal{U}(\mathcal{L}(G))$ such that $\theta = ad(u) \}$

THM Let G icc such that $\mathcal{L}(G)$ does not have prop GAMMA of Murray-von Neumann. TFH:

i) (Phillips '87) $\dfrac{w/nn(C_r^*(G))}{Inn(C_r^*(G))}$ is uncountable

ii) (C-(Diaz-Arias) '19) $w/nn(C_r^*(G)) \trianglelefteq Aut(C_r^*(G))$ is a normal subgroup.

$$\rightsquigarrow \quad sOut(C_r^*(G)) = \frac{Aut(C_r^*(G))}{w/nn(C_r^*(G))}$$

COR For any $G \in SU\mathcal{T}$ we have that

$$sOut(C_r^*(G)) = char(G) \rtimes Out(G).$$

SOME IDEAS BEHIND THE PROOFS:

$$G = K_{*\varphi}, \quad K = K_1 \times K_2 \in \mathcal{T}_0, \quad A \leq K$$
$$QN_G^{(1)}(A) = A$$

$M = \mathcal{L}(G) = \mathcal{L}(H)$ $\quad H \leftarrow$ arbitrary

I) $\Delta : M \to M \bar{\otimes} M$ $\quad \Delta(v_h) = v_h \otimes v_h \quad h \in H$

$\Delta(L(K_1)), \Delta(L(K_2)) = \Delta(L(K)) \subseteq M \bar{\otimes} L(K_{*\varphi})$

commuting non-amen vN algebra

using classification techniques (C-Houdayer 08, Fima-Vaes '10) + (Ioana-Peterson-Popa '05) we get that

"$u\Delta(L(K))u^* \subseteq M \bar{\otimes} L(K)$" (just on a corner)

But let's cheat and assume that

II $\quad \Delta(\mathcal{L}(K)) \subseteq M \otimes \mathcal{L}(K) \qquad \mathcal{L}(H)$

let $k \in K$ and $u_k = \sum_{h \in H} \tau(u_k v_{h^{-1}}) v_h$

$$\sum_{h \in H} \tau(u_k v_{h^{-1}}) v_h \otimes v_h = \Delta(u_k) = E_{M \otimes \mathcal{L}(K)}(\Delta(u_k))$$
$$= \sum_h \tau(u_k v_{h^{-1}}) v_h \otimes E_{\mathcal{L}(K)}(v_h)$$

$$\tau(u_k v_{h^{-1}})\, v_h = \tau(u_k v_{h^{-1}})\, E_{\mathcal{L}(K)}(v_h)$$

$$\tau(u_k v_{h^{-1}}) \neq 0 \implies v_h = E_{\mathcal{L}(K)}(v_h) \in \mathcal{L}(K)$$

$P := \{ h \in H \,/\, v_h \in \mathcal{L}(K)\} \leq H$ - subgroup!

$\implies \mathcal{L}(K) = \mathcal{L}(P) \implies u_g = \eta(g) v_{\delta(g)},\ g \in K$

If cheating is not allowed, quite involved technically; usage of ultrapowers to construct commuting subgroups in H and "bump it up" to a maximal subgroup; etc... (there will be dragones...)

III $\quad u_{\varphi(g)} u_t u_{g^{-1}} = u_{\varphi(g) t g^{-1}} = u_t \quad \forall g \in K$

$$c_g\, v_{\delta(\varphi(g))}\, u_t\, v_{\delta(g^{-1})} = u_t$$

Fourier expansion in $\mathcal{L}(H)$

$$\sum_{h \in H} c_g\, \tau(u_t v_{h^{-1}})\, v_{\delta(\varphi(g)) h \delta(g^{-1})} = \sum_h \tau(u_t v_{h^{-1}}) v_h$$

$$\tau(u_t v_{h^{-1}}) \neq 0 \iff \{ \delta(\varphi(g)) h \delta(g^{-1}) \,/\, g \in B\} \text{ is finite}$$

$$\Updownarrow$$

$$\delta\varphi(g) = h\, \delta_{g}\, h^{-1} \quad \forall g \in B_0 \leq_f B \,.$$

For such $h_1, h_2 \quad \exists\, B_1 \leq_f B$ s.t.

$$\delta(\varphi(g)) = h_1 \delta_{g} h_1^{-1} = h_2 \delta_{g} h_2^{-1} \Rightarrow h_2 h_1^{-1} \in C_H(\delta(B_1))$$

$$\subset C_B(\delta(B_1)) = 1 \quad (\text{icc})$$

self commensurate

There is an unique $h \in H$ such that $u_t = c_t u_h$

OPEN PROBLEMS

Find examples of property (T) groups G such that

- G is W^*-superrigid (Connes 80)
- G is C^*-superrigid
- $\mathrm{Out}(G) = \mathrm{Char}(G) \rtimes \mathrm{Out}(G)$ (Jones 2000)

cI

c

$$C^*_r\big((\Gamma_1 *_{\Sigma} \Gamma_2) \ltimes_B K\big) \overset{\theta}{\cong} C^*_r(H)$$

Σ

$K_1 \times K_2$ $\quad K_i$ = hyp + prop (T)