GOALS 2020

"Duality as the bridge between C^*- and W^*-algebras."

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C^*-algebras		W^*-algebras
$\mathcal{K}(H)$ "compacts"	$\overset{\text{w.o.t. dense}}{\subseteq}$	$B(H)$
$C([0,1])$	$\overset{\text{wk-* dense}}{\subseteq}$	$L^\infty([0,1]) = L^1[0,1]^*$
$\not\cong$		$\cong$
$C(\mathbb{D})$	$\subseteq$	$L^\infty(\mathbb{D})$
$\not\cong$		$\cong$
$C(\mathbb{T}^2)$	$\subseteq$	$L^\infty(\mathbb{T}^2)$

Delicate flowers, Massive beasts

Q: What's the connection?

A: Banach space duality

Thm: If A is a C^*-algebra, then A^{**} is a W^*-algebra

Cor: If $\{a_n\} \subseteq A$, $a \in A$ and $a_n \longrightarrow a$ in A^{**}, then a convex combination of a_n's converges to a in norm.

Duality

Given A, $A^* =$ Banach dual

$= \{$ bdd linear functionals $A \to \mathbb{C}\}$

Examples

① Given $x \in [0,1]$, define

$\varphi_x : C[0,1] \to \mathbb{C}$

$\varphi_x(f) = f(x)$

Q: Under embedding $C[0,1] \subseteq L^\infty([0,1])$, how to extend?

② Given $g \in L^1[0,1]$, define

$\varphi_g : L^\infty([0,1]) \to \mathbb{C}$

$$\varphi_g(f) = \int_{[0,1]} f g \, dx$$

Q: Is there $g \in L^1[0,1]$

s.t. $\varphi_g \Big|_{C[0,1]} = \varphi_x$

In general, if $M \subseteq B(H)$ is W^*-alg. then M has a predual

$M_* = \{$ linear functionals w.o.t.-continuous on ball of $M \}$

e.g. Given $v \in H$,

$$\varphi_v(T) = \langle Tv, v \rangle$$

$$\varphi_v \in M_*$$

Thm $M = (M_*)^*$ and

M_* is unique!

Universal Representation

Notation $S(A) = \{\text{states on } A\}$

$= \{\text{positive elements in } A^* \text{ w/ } \|\cdot\| = 1\}$

Fact $A^* = \operatorname{span} S(A)$

Notation Given $\varphi \in S(A)$, let

$L^2(A, \varphi) =$ GNS Hilbert space

$\pi_\varphi : A \longrightarrow B(L^2(A, \varphi))$

Def^n The universal representation is

$$\pi_{\mathcal{U}} = \bigoplus_{\varphi \in S(A)} \pi_\varphi : A \to B\left(\bigoplus_{\varphi \in S(A)} L^2(A,\varphi)\right)$$

$$L^2(A,\varphi_1) \oplus L^2(A,\varphi_2) \oplus \cdots$$

$$\pi_{\mathcal{U}}(a) = \begin{pmatrix} \pi_{\varphi_1}(a) & 0 & 0 & \\ 0 & \pi_{\varphi_2}(a) & 0 & \\ 0 & 0 & \pi_{\varphi_3}(a) & \\ & & & \ddots \end{pmatrix}$$

Always faithful!

$$\therefore A \cong \pi_{\mathcal{U}}(A) \le \pi_{\mathcal{U}}(A)''$$

<u>Defⁿ</u> $\tilde{\pi}_{u}(A)''$ is universal W^*-alg. of A.

<u>Exercise</u> Every separable rep. of A is a subrep. of $\tilde{\pi}_{u}$.

<u>Thm</u>: $[\tilde{\pi}_{u}(A)'']_* = A^*$

$$\therefore\ A^{**} = \left[\tilde{\pi}_{u}(A)''_*\right]^* = \tilde{\pi}_{u}(A)''$$

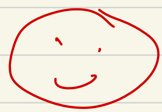
<u>Idea</u>: $\varphi \in S(A)$, the vector state $\langle \cdot\, v_\varphi, v_\varphi \rangle \in [\tilde{\pi}_{u}(A)'']_*$ extends to $A^* \longrightarrow [\tilde{\pi}_{u}(A)'']_*$. $\boxed{\text{QED}}$

Nuclearity and Injectivity

Def^n A is *nuclear* if it can be "linearly approximated by matrices", i.e., $\exists$ diagrams

$$\begin{tikzcd} A \arrow[rr, "id"] \arrow[dr, dashed, "\varphi_n"'] & & A \\ & F_n \arrow[ur, dashed, "\psi_n"'] & \end{tikzcd}$$

$\cancel{\psi_n, \varphi_n = e}$



Def^n $M \subseteq B(H)$ is *injective* if $\exists$ contractive linear map $\Phi : B(H) \to M$

$$\Phi(m) = m \quad \forall m \in M.$$

<u>Thm</u> T.F.A.E.

① A is nuclear

② A^{**} is injective

③ A^{**} is "linearly approximated by matrices in wk-* topology" (semidiscrete)

<u>Cor:</u> If A is nuclear and $J \trianglelefteq A$, then (A/J) is nuclear too!

<u>Proof</u> $A^{**} = (J)^{**} \oplus (A/J)^{*}$.

$\boxtimes$