# Quantum Groups:
# What are they and what are they good for?

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# The Plan

- Introduce the notion of a compact quantum group from an operator algebraic perspective.
- Highlight some examples and illustrate some aspects of their general theory.

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- Compact groups are compact spaces with some extra structure (continuous group law etc...)
- How do we get a non-commutative formulation of compact groups?

# Compact Quantum Groups

## Definition (Woronowicz)

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## Proposition

*If $\mathbb{G} = (A, \Delta)$ is a CQG with* **abelian** *$A$. Then $\exists!$ compact group $G$ so that*

1. $A = C(G)$
2. $(\Delta f)(s, t) = f(st) \qquad (f \in C(G),\ s, t \in G).$

# Sketch

Given $\mathbb{G} = (A, \Delta)$, $(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta$ + cocancellation.

- **By Gelfand**: If $A$ is abelian, then $A = C(G)$ for some unique compact Hausdorff space $G$, moreover the morphism $\Delta : C(G) \to C(G) \otimes_{\min} C(G) \cong C(G \times G)$ comes from a unique continuous map

  $$m : G \times G \to G; \quad (s,t) \mapsto st, \quad \Delta f(s,t) = (f \circ m)(s,t) = f(st).$$

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$$[(\text{id} \otimes \Delta)\Delta]f = [(\Delta \otimes \text{id})\Delta]f \qquad (f \in C(G))$$

$$\iff f(r(st)) = f((rs)t) \qquad (f \in C(G),\ r, s, t \in G)$$

$$\iff m : G \times G \to G \text{ is } \textbf{associative}$$

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- **By cocancellation**: $G$ has left/right cancellation property:

$$\{st = rt\ \forall t \implies s = r\} \quad \& \quad \{ts = tr\ \forall t \implies s = r\}$$

# Basic Examples of Compact Quantum Groups

**Standard Notation:** Given **any** CQG $\mathbb{G} = (A, \Delta)$, we typically write $A = C(\mathbb{G})$. Morally, we think of $C(\mathbb{G})$ as the "C*-algebra of continuous functions on $\mathbb{G}$".

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### Example (Pontryagin duals of discrete groups)

Let $\Gamma$ be a **discrete group**. Put $A = C^*(\Gamma)$, and define

$$\Delta(\gamma) = \gamma \otimes \gamma \qquad (\gamma \in \Gamma \subset \mathbb{C}[\Gamma] \subset A).$$

**Fact**: $\Delta$ linearly and continuously extends to a **comultiplication** $\Delta : A \to A \otimes_{\min} A$ with the cocancellation property.

$\implies$ Get a CQG $\hat{\Gamma} = (C^*(\Gamma), \Delta)$, the Pontryagin Dual of $\Gamma$.

**Note**: When $\Gamma$ is abelian, $\hat{\Gamma}$ is *exactly* the Pontryagin dual of $\Gamma$.

# $q$-deformed $SU(2)$

Here's a first example that doesn't come from groups: $q$-deformed $SU(2)$ quantum group.

- Fix $q \in [-1, 1] \backslash \{0\}$.
- Define a universal C$^*$-algebra $C(SU_q(2))$ with generators $\alpha, \gamma \in C(SU_q(2))$ and relations making the matrix

$$u = \begin{bmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{bmatrix}$$ **unitary** in $M_2(C(SU_q(2)))$ : $u^*u = uu^* = 1.$

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- $C(SU_q(2))$ admits a comultiplication $\Delta$ given by

$$\Delta(u_{ij}) = \sum_{k=1}^{2} u_{ik} \otimes u_{kj}, \quad (u_{ij} = (i,j)\text{th entry of } u).$$

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- (**Woronowicz**) Get a CQG $SU_q(2) = (C(SU_q(2)), \Delta)$, "q-deformed $SU(2)$".

# $q$-deformed $SU(2)$, continued...

$$C(SU_q(2)) = C^*\left(\alpha, \gamma \,\middle|\, u = [u_{ij}] = \begin{bmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{bmatrix} \ \textbf{unitary}\right)$$

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- When $q = 1$, $C(SU_1(2)) = C(SU(2))$. In fact, the generators $\{u_{ij}\}_{i,j=1}^2 = \{\alpha, \gamma\}$ can be identified with the standard coordinate functions on $SU(2)$, and $\Delta$ comes from the "usual" group law:

$$\Delta(u_{ij})(s,t) = \sum_k (u_{ik} \otimes u_{kj})(s,t) = \sum_k u_{ik}(s) u_{kj}(t) = u_{ij}(st).$$

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- The above $q$-deformation procedure $SU(2) \mapsto SU_q(2)$ is a special case of the very general Drinfeld-Jimbo $q$-deformations $G \mapsto G_q$ ($q \in (0,1]$, $G$ cpt. simply conn. s.-simple Lie gp).

# Generalizing $SU_q(2)$: Compact matrix quantum groups

The construction of $SU_q(2)$ as a "non-commutative version of C(SU(2))" can be formalized in the language of compact matrix quantum groups.

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- (For free): The conjugate matrix $\bar{u} = [\overline{u_{ij}}] = [u_{ij}^*] \in M_n(C(G))$ is **invertible**. (It's the conjugate of the representation $u$!)

# Compact matrix quantum groups

## Definition

A compact matrix quantum group (CMQG) is a pair $(A, u)$, where $A$ is a unital C*-algebra and $u = [u_{ij}] \in M_n(A)$ satisfies:

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2. $A$ is **generated** as a C*-algebra by the entries of $u$.
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**Upshot**: Gives a practical way to construct CQGs using mainly "algebraic" generators and relations data.

# Liberations of Matrix Lie Groups

Using the CMQG formalism, we can define "**free versions**" of the classical matrix Lie groups like $G = U_n, O_n, S_n, \ldots$.

**Basic strategy**:

1. Take the coordinates $u_{ij} \in C(G)$.
2. They satisfy some algebraic relations $R_G$ comming from $G$, which includes **commutation**.
3. "Liberate" $G$ by throwing away the commutation relation. Define

$$C(G^+) = C^*\big(u_{ij}, 1 \leq i, j \leq n \;\big|\; R_G \backslash \{\text{commutation}\}\big)$$

4. In nice situations, get a new CMQG $G^+ = (C(G^+), u)$, called the **free version of $G$**.

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In short, we want to "Liberate ~~Michigan!~~ matrix Lie groups!"

# The free unitary quantum group $U_n^+$

Let $n \geq 2$ and let $C(U_n^+)$ be the universal C*-algebra with generators $\{u_{ij}\}_{1 \leq i,j \leq n}$ with the relations

$u = [u_{ij}] \in M_n(C(U_n^+)) \quad \& \quad \bar{u} = [u_{ij}^*] \in M_n(C(U_n^+))$ are **unitary**.

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- $C(U_n^+)$ is the free version of $C(U_n)$: $C(U_n)$ is the **abelianization** of $C(U_n^+)$.
- The formula

$$\Delta u_{ij} = \sum_k u_{ik} \otimes u_{kj}$$

defines a comultiplication $\Delta : C(U_n^+) \to C(U_n^+) \otimes_{\min} C(U_n^+)$.
**Stetch**:

$$u \;\&\; \bar{u} \text{ unitary} \implies [\sum_k u_{ik} \otimes u_{kj}] \;\&\; [\sum_k u_{ik}^* \otimes u_{kj}^*] \text{ unitary}$$

$$\implies \Delta \text{ well-defined by universality!.}$$

$U_n^+ = (C(U_n^+), u)$ is a CMQG, the free unitary quantum group.

# Aside: $U_n^+$ vs. Brown's Algebras

There is a related (older) liberation of the unitary groups $U_n$, due to L. Brown: Define

$$\mathcal{B}_n = C^*\Big(u_{ij}, 1 \leq i,j \leq n \mid u = [u_{ij}] \in M_n(\mathcal{B}_n) \text{ is unitary}\Big).$$

- So $C(U_n^+)$ is a **quotient** of $\mathcal{B}_n$ by the relation "$\bar{u}$ is unitary".
- $\mathcal{B}_n$ has a comultiplication $\Delta : \mathcal{B}_n \to \mathcal{B}_n \otimes_{\min} \mathcal{B}_n$ given by

$$\Delta u_{ij} = \sum_k u_{ik} \otimes u_{kj}.$$

- Is $(\mathcal{B}_n, u, \Delta)$ a compact matrix quantum group?

**Answer**: No! For $\mathcal{B}_n$, $\bar{u}$ is **not invertible**. (Exercise: Try to find explicit unitaries $u \in M_n(B(H))$ such that $\bar{u}$ is not invertible).

# Free orthogonal and permutation quantum groups

We can play the same game as for $U_n$ to get liberations of the orthogonal groups and permutation groups:

Define

$$C(O_n^+) = C^*\Big(v_{ij}, 1 \leq i,j \leq n \mid v = [v_{ij}] \text{ is unitary } \& v_{ij}^* = v_{ij}\Big)$$
$$C(S_n^+) = C^*\Big(p_{ij}, 1 \leq i,j \leq n \mid p = [p_{ij}] \text{ is unitary } \& p_{ij}^2 = p_{ij} = p_{ij}^*\Big).$$

These algebras admit comultiplications
$\Delta : C(G^+) \to C(G^+) \otimes_{\min} C(G^+)$ given by

$$\Delta(x_{ij}) = \sum_k x_{ik} \otimes x_{kj} \qquad (x \in \{v, p\}).$$

Get two new CMQGs: The free orthogonal quantum groups $O_n^+$ and the free permutation quantum groups $S_n^+$.

# The Haar State

**Recall**: Every compact group $G$ admits a unique translation-invariant Borel probability measure $\mu$, called the Haar measure:

$$\int_G f(st)d\mu(t) = \int_G f(ts)d\mu(t) = \int_G f(t)d\mu(t) \qquad (f \in C(G),\ s \in G).$$

Equivalently, if $h : C(G) \to \mathbb{C}$, $h(f) = \int_G f d\mu$ is the corresponding **state**, then

$$(\text{id} \otimes h)\Delta(f) = (h \otimes \text{id})\Delta(f) = h(f)1 \qquad (f \in C(G)).$$

## Definition

A Haar state on a CQG $\mathbb{G} = (C(\mathbb{G}), \Delta)$ is a state $h : C(\mathbb{G}) \to \mathbb{C}$ satisfying

$$(\text{id} \otimes h)\Delta(x) = (h \otimes \text{id})\Delta(x) = h(x)1 \qquad (x \in C(\mathbb{G})).$$

**Example**: On a **group dual** $\hat{\Gamma} = (C^*(\Gamma), \Delta)$, the Haar state is given by the canonical group trace $h(\gamma) = \tau_\Gamma(\gamma) = \delta_{\gamma,e}$. (Check!)

# Existence and Uniqueness of Haar State

## Theorem (Woronowicz)

*Every CQG $\mathbb{G}$ admits a unique Haar state $h$. (which could be non-tracial or non-faithful).*

**Sketch**:

- Given linear functionals $\varphi, \psi \in C(\mathbb{G})^*$, define their convolution product $\varphi \star \psi \in C(\mathbb{G})^*$ by
$$\varphi \star \psi = (\varphi \otimes \psi)\Delta.$$
Coassociativity of $\Delta$ makes $(C(\mathbb{G})^*, \star)$ into a **Banach algebra**, and the state space $S(C(\mathbb{G}))$ is a subsemigroup.
- $h \in S(C(\mathbb{G}))$ is a Haar state if and only if
$$\varphi \star h = h \star \varphi = h \qquad \varphi \in S(C(\mathbb{G})).$$
- Start with any **faithful state** $\psi \in S(C(\mathbb{G}))$, consider the weak$^*$-limit
$$h := \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \psi^{\star k} \in S(C(\mathbb{G})) \implies h \textbf{ is a Haar state!}.$$

# Application: Quantum Group Operator Algebras

Using the Haar state, we can form analogues of our favorite group operator algebras:

- Do the GNS construction: Let $L^2(\mathbb{G}) = L^2(C(\mathbb{G}), h)$ and $\lambda : C(\mathbb{G}) \to B(L^2(\mathbb{G}))$ be the associated "left-regular representation".
- Get the reduced C*-algebra of $\mathbb{G}$:

$$C_r(\mathbb{G}) := \lambda(C(\mathbb{G})) \subseteq B(L^2(\mathbb{G})).$$

- Get the von Neumann algebra of $\mathbb{G}$:

$$L^\infty(\mathbb{G}) = \lambda(C(\mathbb{G}))'' \subseteq B(L^2(\mathbb{G})).$$

- One can also construct a universal C*-algebra of $\mathbb{G}$, $C^u(\mathbb{G})$: If $\mathbb{G} = (C(\mathbb{G}), u = [u_{ij}])$ is a CMQG, put

$$\mathcal{O}(\mathbb{G}) = * - \text{alg}\Big(u_{ij}, 1 \leq i, j \leq n\Big).$$

**Fact:** Haar state $h$ is always faithful on $\mathcal{O}(\mathbb{G})$. Then define

$$C^u(\mathbb{G}) = C^*_{\text{univ.}}(\mathcal{O}(\mathbb{G})).$$

# Application: Quantum Group Operator Algebras

The algebras $C^u(\mathbb{G}), C_r(\mathbb{G})$ and $L^\infty(\mathbb{G})$ simultaneously generalize

1. The algebras of multiplication operators $C(G), L^\infty(G) \subseteq B(L^2(G))$ on compact groups $G$.
2. The discrete group operator algebras $C^*(\Gamma), C_r^*(\Gamma), L\Gamma$.

Can even generalize the notion of **amenability**:

### Definition

A CQG $\mathbb{G}$ is coamenable iff the canonical quotient map $C^u(\mathbb{G}) \to C_r(\mathbb{G})$ is injective.

- Coamebility $\implies$ $C_r(\mathbb{G})$ nuclear, $L^\infty(\mathbb{G})$ injective.
- $SU_q(2)$ is coamenable.
- Liberations $G^+$ of matrix Lie groups are generally not coamenable. $\implies$ $C_r(G^+), L^\infty(G^+)$ are interesting!

### Examples (Many hands)

$L^\infty(U_2^+) \cong L(\mathbb{F}_2)$. $L^\infty(G^+)$ is a full, strongly solid, weakly amenable, A-T-menable $\mathrm{II}_1$-factor. $C_r(G^+)$ is simple with unique trace, ... (for $G^+ = U_n^+, O_n^+, S_n^+$).

# Other Applications, briefly

Compact quantum groups can be used for many other non-commutative purposes:

1. $L^\infty(\mathbb{G})$ can be regarded as a non-commutative **probability space** with respect to the Haar state (See Ian's talk).
2. CQGs can be made to **act** on various mathematical structures: graphs, metric spaces, OAs, subfactors.
3. CQG's often appear as symmetries in quantum information theory, free probability etc.
4. CQG's have a rich representation theory - give rise to many interesting examples of **rigid C$^*$-tensor categories** (see Corey's talk. )

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Thanks!