# FREE PROBABILITY OF TYPE B AND ASYMPTOTICS OF FINITE-RANK PERTURBATIONS OF RANDOM MATRICES

Dima Shlyakhtenko, UCLA

Free Probability and Large $N$ Limit, V

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# Wigner's Semicircle Law

Let $A(N)$ be an $N \times N$ random matrix so that

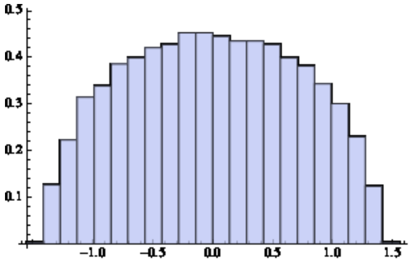
$$\{\mathrm{Re}(A_{ij}), \mathrm{Im}(A_{ij}) : 1 \le i < j \le N\} \cup \{A_{kk} : 1 \le k \le N\}$$

are iid real Gaussians of variance $N^{-1/2}(1+\delta_{ij})$. Let $\lambda_1^A(N) \le \cdots \le \lambda_N^A(N)$ be the eigenvalues of $A(N)$, and let

$$\mu_N^A = \frac{1}{N}\sum_j \delta_{\lambda_j^A(N)}.$$

Then as $N \to \infty$

$$\mathbb{E}[\mu_N^A] \to \text{semicircle law} = \frac{1}{\pi}\sqrt{2-t^2}\chi_{[-\sqrt{2},\sqrt{2}]}dt$$



# Voiculescu's Asymptotic Freeness

$A(N)$ as before, $B(N)$ diagonal matrix with eigenvalues $\lambda_1^B(N) \leq \cdots \leq \lambda_N^B(N)$. Assume that

$$\mu_N^B = \frac{1}{N} \sum_j \delta_{\lambda_j^B(N)} \to \mu^B.$$

Then $A(N)$ and $B(N)$ are asymptoically freely independent. In particular,

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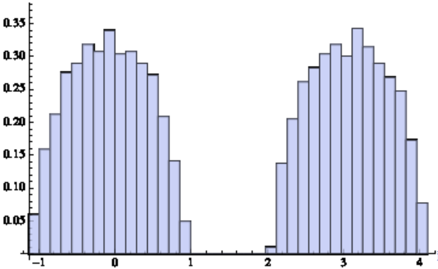
Example: $B = 3P_N$ with $P_N$ a projection of rank $N/2$.



# Analytic Subordination and Free Convolution [Biane,Voiculescu,...]

To compute $\eta = \mu^A \boxplus \mu^B$ define $G_\nu = \int \frac{1}{z-t} d\nu(t)$. Then there exist analytic functions $\omega_A, \omega_B : \mathbb{C}^+ \to \mathbb{C}^+$ uniquely determined by

- $G_{\mu^A}(\omega_A(z)) = G_{\mu^B}(\omega_B(z)) = G_\eta(z)$
- $\omega_A(z) + \omega_B(z) = z + 1/G_\eta(z)$
- $\lim_{y\uparrow\infty} \omega_A(iy)/(iy) = \lim_{y\to\infty} \omega_A'(iy) = 1$ and same for $\omega_B$.

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Put $F_\nu(z) = 1/G_\nu(z)$. Then $R_\nu(z) = F_\nu^{-1}(z) + z$ so that $R_{\mu^A}(z) + R_{\mu^B}(z) = R_{\mu^A \boxplus \mu^B}(z)$ becomes

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# Finite-rank perturbations [Ben Arous, Baik, Peche].

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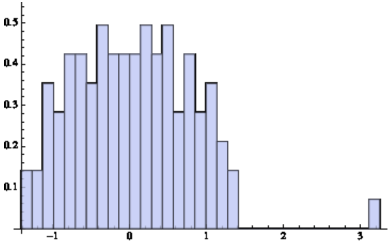
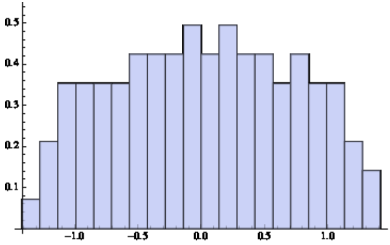
Let $A(N)$ be as before but consider $B(N)$ a finite rank matrix (e.g. $B_N = \theta Q_N$) with $Q_N$ rank 1 projection.
Semicircular limit for $A(N) + B(N)$ but there may or may not be outlier eigenvalues:




$\theta = 3$




$\theta = 0.5$

# Finite rank perturbations and freeness?

It was discovered (Capitaine, Belischi-Bercovici-Capitain-Fevrier) that the description of the outlier involves free subordination functions. For example, if $A^N$ is GUE and $B^N$ has 1 eigenvalue $\theta$ and the rest zero, then we set

$$(\omega_A, \omega_B) = \text{subordination functions for } \eta \boxplus \delta_0$$

with $\eta = \text{semicircle law}$, i.e., $\omega_A(z) = F_\eta^{-1}(z)$, $\omega_B(z) = z$, then there will be an outlier at $\theta' = \omega_A(\theta)$ (i.e. $G_\mu(\theta') = 1/\theta$).

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with $\eta = \text{semicircle law}$, i.e., $\omega_A(z) = F_\eta^{-1}(z)$, $\omega_B(z) = z$, then there will be an outlier at $\theta' = \omega_A(\theta)$ (i.e. $G_\mu(\theta') = 1/\theta$). Why?! Is there still some free independence involved?

# Another look at laws of random matrices

We consider the $1/N$ expansion of the law of $A^N + B^N$ :

$$\mu_N^{A+B} = \mu^{A+B} + \frac{1}{N}\dot{\mu}^{A+B} + o(N^{-1}).$$

The idea is that moving 1 eigenvalue out of $N$ gives a perturbation of $\mu^{A+B}$ which is of order $1/N$. Our aim is to compute $\dot{\mu}^{A+B}$. Thus we want to keep track of the pair $\mu^{A+B}, \dot{\mu}^{A+B}$ and not just $\mu^{A+B}$ (ordinary free probability).

# Infinitesimal free probability theory [Belinschi-D.S, 2012]

To encode such questions we consider an infinitesimal probability space $(A, \phi, \phi')$ where $A$ is a unital algebra, $\phi, \phi' : A \to \mathbb{C}$ are linear functionals and $\phi(1) = 1$, $\phi'(1) = 0$.

Example

Let $(A, \phi_t)$ be a family of probability spaces, and assume that $\phi_t = \phi + t\phi' + o(t)$. Then $(A, \phi, \phi')$ is an infinitesimal probability space.

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Eg: $X_t$ family of random variables and you define $\phi_t : \mathbb{C}[t] \to \mathbb{C}$ by $\phi_t(p) = \mathbb{E}(p(X_t))$.

# Infinitesimal freeness, ctd.

We say that $A_1, A_2 \subset A$ are infinitesimally free if the freeness condition in $(A, \phi_t = \phi + t\phi')$ holds to order $o(t)$.

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In other words, the following conditions holds whenever $a_1, \ldots, a_r \in A$ are such that $a_k \in A_{i_k}$, $i_1 \neq i_2$, $i_2 \neq i_3$, $\ldots$ and $\phi(a_1) = \phi(a_2) = \cdots = \phi(a_n) = 0$:

$$\begin{aligned} \phi(a_1 \cdots a_r) &= 0; \\ \phi'(a_1 \cdots a_r) &= \sum_{j=1}^{r} \phi(a_1 \cdots a_{j-1} \phi'(a_j) a_{j+1} \cdots a_r). \end{aligned}$$

# Free probability of type B [Biane-Goodman-Nica, 2003]

We introduced infinitesimal free probability theory to get a better understanding of type B free probability introduced by Biane-Goodman-Nica. Their motivation was purely combinatorial: free probability is obtained from classical probability by replacing the lattice of all partitions by the lattice of (type A) non-crossing partitions:

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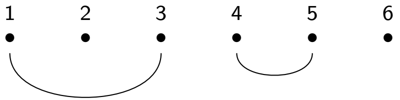

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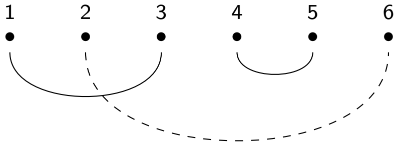

# Non-crossing partitions of type B

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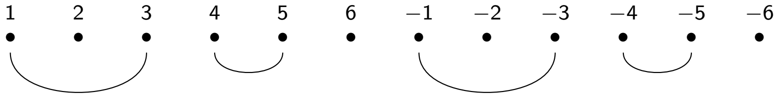

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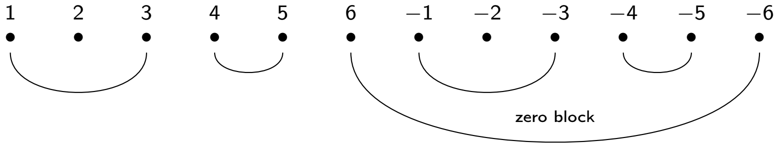

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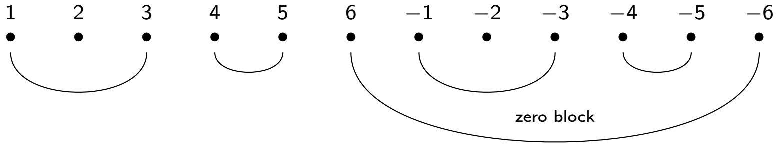


Type A non-crossing partitions have to do with geodesics in $(S_n, \text{transpositions})$ connecting 1 and $(1 \dots n)$.

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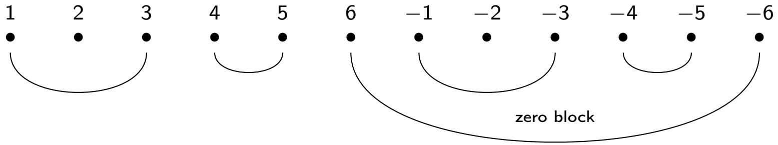


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Type B: same for the hyperoctahedral group.

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## Theorem (Belinschi+DS '12)

*Let $(\mu_1, \mu_1')$ and $(\mu_2, \mu_2')$ be infinitesimal laws: $\mu_j$ measures and $\mu_j'$ distributions satisfying certain conditions. Let $X_j(t) \in (A, \phi)$ so that $\mu^{X_j(t)} \sim \mu_j + t\mu_j' + O(t^2)$, and assume $X_1(t), X_2(t)$ are free for all $t$. Then $Y(t) = X_1(t) + X_2(t) \sim \eta + t\eta' + O(t^2)$ where:*

- $\eta = \mu_1 \boxplus \mu_2$
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Can also consider multiplicative convolution $\boxtimes_B$ etc.

# Asyptotic infinitesimal freeness

## Theorem

*Let $A(N)$ be a Gaussian random matrix and let $B(N)$ be a finite-rank matrix. Let $\tau_N$ be the joint law of $A(N)$ and $B(N)$ with respect to $N^{-1}Tr$. Then $\tau_N = \tau + \frac{1}{N}\tau' + o(N^{-1})$ and moreover $A(N)$ and $B(N)$ are infinitesimally free under $(\tau, \tau')$.*

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## Corollary

*Let $\mathcal{A}(N), \mathcal{B}(N) \in (A, \phi)$ be operators having the same law of $A(N)$ and $B(N)$ respectively, but such that $\mathcal{A}(N)$ and $\mathcal{B}(N)$ are free for each $N$. Then*

$$\mu^{\mathcal{A}(N)+\mathcal{B}(N)} = \mu^{A(N)+B(N)} + o(1/N).$$

*In particular,*

$$\mu^{A(N)+B(N)} = \mu^{A(N)} \boxplus \mu^{B(N)} + o(1/N)$$

*explaining the connection with free convolution.*

# Example.

Let $B_N = \theta E_{11}$ with $E_{11}$ rank one projection with entry 1 in position $1, 1$ and zero elsewhere.
Then

$$\mu^{B_N} = \delta_0 + \frac{1}{N}(\delta_\theta - \delta_0).$$

If $A_N$ is a Gaussian random matrix and $\eta$ is the semicircle law, then

$$\mu^{A_N} = \eta + O(N^{-2}).$$

So: $\mu^{A_N+B_N} = \mu + \frac{1}{N}\dot{\mu} + o(1/N)$ and

$$(\mu, \dot{\mu}) = (\eta, 0) \boxplus_B (\delta_0, \delta_\theta - \delta_0).$$

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General theory implies:

$$\begin{aligned} G_{\dot\mu}(z) &= \partial_z \int \log(z-t) d\dot\mu(t) \\ &= \partial_z \int \frac{1}{z-t}[h_+(t) - h_-(t)]dz, \quad h_\pm \text{ monotone} \end{aligned}$$

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# Example, ctd.

Formula for $\boxplus_B$ involving subordination functions gives:

$$
\begin{aligned}
G_{\dot{\mu}}(z) &= F'_\mu(z)\left(\frac{1}{F_\mu(z)-\theta} - \frac{1}{F_\mu(z)}\right) = \partial_z \log\left(\frac{F_\mu(z)-\theta}{F_\mu(z)}\right) \\
&= \partial_z \log(1-\theta G_\mu(z))
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# Example, ctd.

Formula for $\boxplus_B$ involving subordination functions gives:

$$\begin{aligned} G_{\dot\mu}(z) &= F'_\mu(z)\left(\frac{1}{F_\mu(z)-\theta}-\frac{1}{F_\mu(z)}\right)=\partial_z\log\left(\frac{F_\mu(z)-\theta}{F_\mu(z)}\right)\\ &= \partial_z\log(1-\theta G_\mu(z))=\partial_z\int(z-t)^{-1}[h_+(t)-h_-(t)]dt. \end{aligned}$$

$$\int\frac{[h_+(t)-h_-(t)]}{z-t}dt=\log\left(1-\theta G_\mu(z)\right)=\log(1-\theta(z-\sqrt{z^2-2})).$$

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If $\theta > 1/\sqrt{2}$: let $\theta'$ be the solution to $G_\eta(\theta') = 1/\theta$. Then

$$\dot{\mu} = \delta_{\theta'} - \frac{\theta(t - 2\theta)}{\pi(2\theta(t - \theta) - 1)\sqrt{2 - t^2}} \chi_{[-\sqrt{2}, \sqrt{2}]} dt$$

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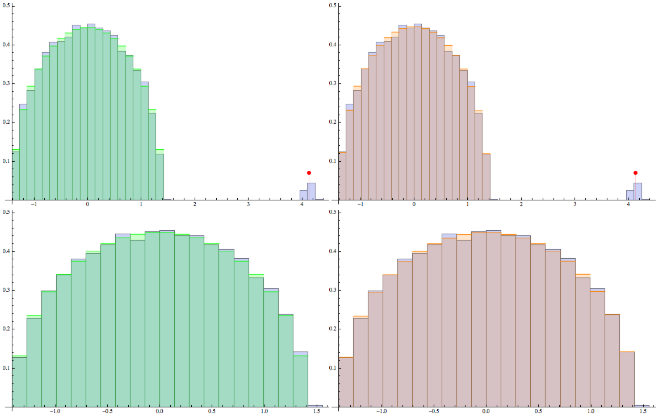
Thus $d\mu_N = \frac{1}{\pi}\sqrt{2 - t^2} \chi_{[-\sqrt{2}, \sqrt{2}]} + \frac{1}{N}\dot{\mu} + O(N^{-2})$.

# Numerical simulation



Average of 40 complex $100 \times 100$ matrices, with $\theta = 4$ or $\theta = 0.4$.

# Ideas of proof

It turns out that $\mu_N^A = \mu + O(1/N^2)$. On the other hand, if $E_{ij}$ is the matrix with 1 in the $i,j$-th entries and zeros elsewhere, then for any fixed $p$, $\frac{1}{N}Tr(p(\{E_{ij}\}) = p(0) + \frac{1}{N}\dot{\tau}(p)$. For example, the law of $\theta E_{11}$ is $\delta_0 + \frac{1}{N}(\delta_\theta - \delta_0)$.

## Lemma

Then for any polynomials $q_1, \ldots, q_r$,

$$\lim_{N\to\infty} \mathbb{E}\,Tr\Big[E_{i_r j_1} q_1(A(N)) E_{i_1 j_2} q_2(A(N)) E_{i_2 j_3} \times \cdots \times E_{i_{r-1} j_r} q_r(A(N))\Big] = \prod_{s=1}^{r} \delta_{j_s = i_s} \tau(q_s) \qquad \text{i.e.}$$

$$\lim_{N\to\infty} \mathbb{E}\,Tr(E_{i_r j_1} q_1 E_{i_1 j_2} q_2 E_{i_2 j_3} \cdots q_r) = \prod_{s=1}^{r} \lim_{N} \mathbb{E}\,Tr(E_{j_s j_s} q_s E_{i_s i_s})$$

Compute or use concentration.

# Infinitesimal freeness is not for free!

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where $(\nu, \nu')$ is an infinitesimal semicircle law ($\nu$ is semicircular, $\nu' = \text{arcsine} - \text{semicircular}$). But computation shows [Johannsen] that $\eta' = \frac{1}{4}(\delta_{\sqrt{2}} + \delta_{-\sqrt{2}}) - \frac{1}{2\pi}\frac{1}{\sqrt{1-t^2}}\chi_{[-\sqrt{2},\sqrt{2}]}dt$ is not the arcsine law.

# Remarks

- Same statement holds if we assume that $A(N) = U(N)D(N)U(N)^*$ with $U(N)$ Haar-distributed unitary matrix and $D(N)$ a diagonal matrix so that $\mu_N^D$ are all supported on a compact set and $\mu_N^D \to \mu^D$ weakly.
- Can also handle the real Gaussian case, which is different in that $\mu_N^A = \eta + \frac{1}{N}\dot{\eta} + o(1/N)$ with $\eta$ the semicircle law and

$$\begin{aligned} \dot{\eta} &= \frac{1}{4}(\delta_{\sqrt{2}} + \delta_{-\sqrt{2}}) - \frac{1}{2\pi\sqrt{2-t^2}}\chi_{[-\sqrt{2},\sqrt{2}]}(t)dt \\ \dot{\mu}_{\text{real}} &= \dot{\mu}_{\text{complex}} + \dot{\eta}. \end{aligned}$$

- We can also deduce formulas for other polynomials in $A(N)$ and $B(N)$, such as products $B(N)A(N)^2B(N)$.

Thank you!