Local law of addition of random matrices

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[1]Supported by ERC Advanced Grant RANMAT No. 338804

Spectrum of sum of random matrices

Question: Given $A = \mathrm{diag}\,(a_1, \ldots, a_N)$ and $B = \mathrm{diag}\,(b_1, \ldots, b_N)$, what is the eigenvalue density of the random matrix

$$H = A + UBU^*$$

if U is a Haar unitary and N is large?

Answer: [Voiculescu '91]

Let $$\mu_A := \frac{1}{N}\sum_{i=1}^{N}\delta_{a_i}, \qquad \mu_B := \frac{1}{N}\sum_{i=1}^{N}\delta_{b_i}.$$

Then for large N the empirical spectral distribution of $A + UBU^*$,

$$\mu_H := \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i}\,, \qquad \lambda_i \ : \text{eigenvalues of } H\,,$$

is close to $\mu_A \boxplus \mu_B$, the free additive convolution of μ_A and μ_B.

Of course, we choose neither A nor B to be multiples of the identity matrix.
Wlog: $\mathrm{Tr}A = \mathrm{Tr}B = 0$.

Stieltjes transform

Definition: For any probability measure ν, its Stieltjes transform $m_\nu(z)$ is defined by

$$m_\nu(z) := \int_{\mathbb{R}} \frac{1}{x-z} \, \mathrm{d}\nu(x) \,, \qquad z \in \mathbb{C}^+.$$

Observe: $m_\nu : \mathbb{C}^+ \to \mathbb{C}^+$, analytic and $\lim_{\eta \nearrow \infty} \mathrm{i}\eta \, m_\nu(\mathrm{i}\eta) = -1$.

Define (negative) reciprocal Stieltjes transform:

$$F_\nu(z) := -\frac{1}{m_\nu(z)} \,, \qquad z \in \mathbb{C}^+.$$

Observe: $F_\nu : \mathbb{C}^+ \to \mathbb{C}^+$, analytic and $\lim_{\eta \nearrow \infty} \frac{F_\nu(\mathrm{i}\eta)}{\mathrm{i}\eta} = 1$.

Free additive convolution

Analytic definition via subordination functions: Symmetric binary operation on the set of probability measures uniquely characterized by the following result:

Theorem (Belinschi-Bercovici '07, Chistyakov-Götze '11).

Given μ_A and μ_B (thus also F_{μ_A} and F_{μ_B}), there exist unique analytic $\omega_A, \omega_B : \mathbb{C}^+ \to \mathbb{C}^+$, such that

(1) $\operatorname{Im}\omega_A(z), \operatorname{Im}\omega_B(z) \geq \operatorname{Im} z$ *and* $\lim_{\eta \nearrow \infty} \frac{\omega_A(\mathrm{i}\eta)}{\mathrm{i}\eta} = \lim_{\eta \nearrow \infty} \frac{\omega_B(\mathrm{i}\eta)}{\mathrm{i}\eta} = 1;$

(2)

$$\left.\begin{aligned} F_{\mu_A}(\omega_B(z)) &= \omega_A(z) + \omega_B(z) - z \\ F_{\mu_B}(\omega_A(z)) &= \omega_A(z) + \omega_B(z) - z \end{aligned}\right\} \text{self-consistent equation (SCE) for } \omega_A,\, \omega_B\,.$$

By (2): $F_{\mu_A}(\omega_B(z)) = F_{\mu_B}(\omega_A(z)) =: F(z)$.

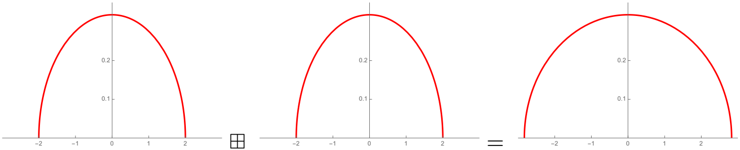
By (1) : $F(z)$ is the reciprocal Stieltjes transform of a probability measure: $\mu_A \boxplus \mu_B$.

Algebraic definition: Addition of free random variables [Voiculescu '86].

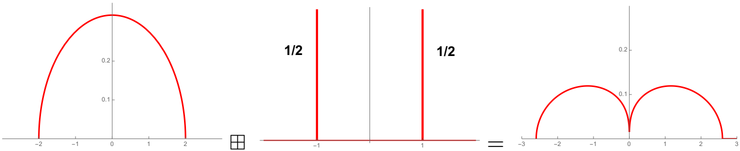
Subordination phenomenon: [Voiculescu '93], [Biane '98].

Examples I

semicircle ⊞ semicircle

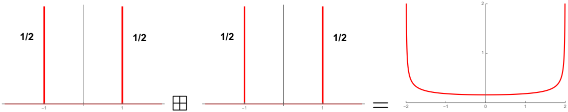


semicircle ⊞ Bernoulli

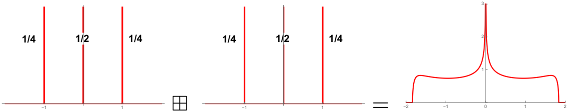


Examples II

Bernoulli ⊞ Bernoulli



three point masses ⊞ three point masses



> Definition:
>
> Regular bulk: Free additive convolution admits a finite and strictly positive density.

Lemma: Inside the regular bulk,

$$\lim_{\eta \searrow 0} \operatorname{Im} \omega_A(E + \mathrm{i}\eta) > 0\,, \qquad \lim_{\eta \searrow 0} \operatorname{Im} \omega_B(E + \mathrm{i}\eta) > 0\,.$$

Theorem (Voiculescu '91).

Let $H = A + UBU^*$ *and* $\mu_H := \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i}$, *with* (λ_i) *the eigenvalues of* H.

For any fixed interval $\mathcal{I} \subset \mathbb{R}$,

$$\frac{|\mu_H(\mathcal{I}) - \mu_A \boxplus \mu_B(\mathcal{I})|}{|\mathcal{I}|} \xrightarrow{a.s.} 0\,, \qquad N \to \infty.$$

Alternative proofs: [Speicher '93], [Biane '98], [Pastur-Vasilchuk '00], [Collins '03],...

Question 1 (local law): Does the convergence still hold if $|\mathcal{I}| = o(1)$, and how small can $|\mathcal{I}|$ be?

Question 2 (convergence rate): What is the convergence rate, as $N \nearrow \infty$, of

$$\sup_{\mathcal{I}\subset\mathbb{R}} \left|\mu_H(\mathcal{I}) - \mu_{A\boxplus B}(\mathcal{I})\right|.$$

Questions 1 and 2 are related.

Main result:

Theorem (Bao-Erdős-S. '15b).

Let $H = A + UBU^*$ *and* $\mu_H := \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i}$, *with* (λ_i) *the eigenvalues of* H.

Fix any $\gamma > 0$. *For any compact interval* $\mathcal{I}$ *in the regular bulk with* $|\mathcal{I}| \geq N^{-1+\gamma}$,

$$\frac{|\mu_H(\mathcal{I}) - \mu_A \boxplus \mu_B(\mathcal{I})|}{|\mathcal{I}|} \prec \frac{1}{\sqrt{N|\mathcal{I}|}},$$

for N *sufficiently large.*

Main result:

Theorem (Bao-Erdős-S. '15b).

Fix any $\gamma > 0$. *For any compact interval* $\mathcal{I}$ *in the regular bulk with* $|\mathcal{I}| \geq N^{-1+\gamma}$, *we have*

$$\frac{|\mu_H(\mathcal{I}) - \mu_A \boxplus \mu_B(\mathcal{I})|}{|\mathcal{I}|} \prec \frac{1}{\sqrt{N|\mathcal{I}|}},$$

for N *sufficiently large.*

Remarks:

- Technical assumption: $\|A\|, \|B\| \leq C$.
- Typical eigenvalue spacing in the regular bulk is order $1/N$.
- Special case: Entries of A and B are supported at two points (Bernoulli).
- Previous results:

$$\frac{|\mu_H(\mathcal{I}) - \mu_A \boxplus \mu_B(\mathcal{I})|}{|\mathcal{I}|} \prec \frac{1}{N|\mathcal{I}|^7}, \qquad |\mathcal{I}| \geq N^{-1/7+\gamma} \qquad \text{[Kargin '12-'15]}$$

$$\frac{|\mu_H(\mathcal{I}) - \mu_A \boxplus \mu_B(\mathcal{I})|}{|\mathcal{I}|} \prec \frac{1}{N|\mathcal{I}|^{3/2}}, \qquad |\mathcal{I}| \geq N^{-2/3+\gamma} \qquad \text{[Bao-Erdős-S. '15a]}$$

Main technical result: Local law

Local law is mostly stated in terms of the Green function $G(z) := (H - z)^{-1}$. Link with Stieltjes transform $m_H \equiv m_{\mu_H}$: $\operatorname{tr} G(z) = \frac{1}{N}\sum_{i=1}^{N} \frac{1}{\lambda_i - z} = m_H(z)\,, \qquad \operatorname{tr} := \frac{1}{N}\operatorname{Tr}.$

Theorem (Bao-Erdős-S. '15b).

Choose any compact interval $\mathcal{I}$ in the regular bulk of $\mu_A \boxplus \mu_B$, and set

$$\mathcal{S}_{\mathcal{I}}(\gamma) := \{z = E + \mathrm{i}\eta : E \in \mathcal{I}\,, N^{-1+\gamma} \le \eta < \infty\}\,.$$

For any (small) $\gamma > 0$, we have

$$\begin{aligned} \Big| m_H(z) - m_{\mu_A \boxplus \mu_B}(z) \Big| &\prec \frac{1}{\sqrt{N\eta}}\,, \\ \Big| G_{ij}(z) - \frac{\delta_{ij}}{a_i - \omega_B(z)} \Big| &\prec \frac{1}{\sqrt{N\eta}}\,, \qquad \textit{uniformly on} \quad \mathcal{S}_{\mathcal{I}}(\gamma)\,. \end{aligned}$$

Recall: $m_{\mu_A \boxplus \mu_B}(z) = m_{\mu_A}(\omega_B(z)) = \frac{1}{N}\sum_{i=1}^{N} \frac{1}{a_i - \omega_B(z)}.$

About local laws in RMT

Local laws for the spectrum of random matrices have been widely studied since the works by Erdős-Schlein-Yau-Yin etc.. It serves as an input for proving the universality of local statistics.

Some reference: (on optimal scale)

- (Wigner type matrices) [Erdős-Schlein-Yau '07-'09], [Tao-Vu '09-'12], [Erdős-Yau-Yin '10-'12], [Erdős-Knowles-Yau-Yin '13], [Ajanki-Erdős-Krüger '15], [Gőtze-Naumov-Tikhomirov '15], ...

Remarks:

- Schur complement is used, which expresses G_{ii} in terms of $\mathbf{a}_i^* G^{(i)} \mathbf{a}_i$, where $\mathbf{a}_i$ is a column of the matrix and $G^{(i)}$ (a submatrix of G) is independent of $\mathbf{a}_i$.

Local stability of SCE

Let $$\Phi_{\mu_A,\mu_B}(\omega_1,\omega_2,z) := \begin{pmatrix} F_{\mu_A}(\omega_2) - \omega_1 - \omega_2 + z \\ F_{\mu_B}(\omega_1) - \omega_1 - \omega_2 + z \end{pmatrix} .$$

SCE for ω_A, ω_B: $$\Phi_{\mu_A,\mu_B}(\omega_A(z),\omega_B(z),z) = 0 .$$

Local Stability: [Bao-Erdős-S. '15a]

Fix $z \in \mathcal{S}_{\mathcal{I}}(\gamma)$. Assume $\omega_A^c, \omega_B^c, \mathbf{r}$ satisfy $\operatorname{Im}\omega_A^c(z), \operatorname{Im}\omega_B^c(z) > 0$ and

$$\Phi_{\mu_A,\mu_B}(\omega_A^c(z),\omega_B^c(z),z) = \mathbf{r}(z) ,$$

and that there is a small $\delta > 0$ such that

$$|\omega_A^c(z) - \omega_A(z)| \le \delta , \qquad |\omega_B^c(z) - \omega_B(z)| \le \delta .$$

Then we have, in the regular bulk, uniformly in $\operatorname{Im} z \ge 0$,

$$|\omega_A^c(z) - \omega_A(z)| \le C\|\mathbf{r}(z)\| , \qquad |\omega_B^c(z) - \omega_B(z)| \le C\|\mathbf{r}(z)\| .$$

Previous results: Local stability with an additional condition [Kargin '13].

Perturbed SCE for random matrix

Approximate subordination functions:

$$\omega_A^c(z) := z - \frac{\mathrm{tr} AG(z)}{m_H(z)}\,, \qquad \omega_B^c(z) := z - \frac{\mathrm{tr} UBU^*G(z)}{m_H(z)}\,.$$

Since $(A + UBU^* - z)G(z) = I$, we have

$$-\frac{1}{m_H(z)} = \omega_A^c(z) + \omega_B^c(z) - z\,.$$

Our aim: Show that

$$\|\Phi_{\mu_A,\mu_B}(\omega_A^c(z), \omega_B^c(z), z)\| \prec \frac{1}{\sqrt{N\eta}}\,, \qquad z = E + \mathrm{i}\eta\,,$$

which is equivalent to

$$m_H(z) = m_{\mu_A}(\omega_B^c(z)) + O_\prec\left(\frac{1}{\sqrt{N\eta}}\right)\,,$$
$$m_H(z) = m_{\mu_B}(\omega_A^c(z)) + O_\prec\left(\frac{1}{\sqrt{N\eta}}\right)\,.$$

Main task: Prove

$$G_{ii}(z) = \frac{1}{a_i - \omega_B^c(z)} + O_\prec\left(\frac{1}{\sqrt{N\eta}}\right).$$

Non-optimal way: Using the full randomness of U at once

Full expectation $\mathbb{E}[G_{ii}]$

$+$

Gromov-Milman concentration for $G_{ii} - \mathbb{E}[G_{ii}]$.

Optimal way: Separating some partial randomness $\mathbf{v}_i$ from U

Partial expectation $\mathbb{E}_{\mathbf{v}_i}[G_{ii}]$

$+$

Concentration for $G_{ii} - \mathbb{E}_{\mathbf{v}_i}[G_{ii}]$.

Remark: Shorthand $\mathbb{E}_i := \mathbb{E}_{\mathbf{v}_i}$. In general, identifying $\mathbb{E}[\cdot]$ is easier than identifying $\mathbb{E}_i[\cdot]$, while estimating $(\mathrm{Id} - \mathbb{E})[\cdot]$ is harder than estimating $(\mathrm{Id} - \mathbb{E}_i)[\cdot]$.

Householder reflection as partial randomness

Proposition (Diaconis-Shahshahani '87).

U Haar distributed on $\mathcal{U}(N)$,

$$U = -\mathrm{e}^{\mathrm{i}\theta_1}(I - 2\mathbf{r}_1\mathbf{r}_1^*)\begin{pmatrix} 1 & \\ & U^1 \end{pmatrix} := -\mathrm{e}^{\mathrm{i}\theta_1} R_1 U^{\langle 1 \rangle}\,,$$

$$\mathbf{r}_1 := \frac{\mathbf{e}_1 + \mathrm{e}^{-\mathrm{i}\theta_1}\mathbf{v}_1}{\|\mathbf{e}_1 + \mathrm{e}^{-\mathrm{i}\theta_1}\mathbf{v}_1\|_2}\,.$$

$\mathbf{v}_1$ denotes the first column of U, $\mathbf{v}_1$ is uniformly distributed on $\mathcal{S}_{\mathbb{C}}^{N-1}$,
U^1 is Haar on $\mathcal{U}(N-1)$,
$\mathbf{v}_1$ and U^1 are independent.

Remark 1: $-\mathrm{e}^{\mathrm{i}\theta_1} R_1$ is the Householder reflection sending $\mathbf{e}_1$ to $\mathbf{v}_1$.

Remark 2: Analogously, we have an independent pair $\mathbf{v}_i$ and U^i for all i.

Remark 3: Independence between $\mathbf{v}_i$ and U^i enables us to work with the partial expectation $\mathbb{E}_{\mathbf{v}_i}[G_{ii}]$.

Concentration of Green function elements

Lemma.

For all $z \in \mathcal{S}_{\mathcal{I}}(\gamma)$,

$$\Big|G_{ii}(z) - \mathbb{E}_i[G_{ii}(z)]\Big| \prec \frac{1}{\sqrt{N\eta}}\,, \qquad\qquad z = E + \mathrm{i}\eta\,.$$

Proof: Use resolvent expansions to write

$$G_{ii} = G^{[i]}_{ii} + \frac{\Psi_i}{\Xi_i}\,,$$

$G^{[i]}$: a matrix independent of $\mathbf{v}_i$;
Ψ_i, Ξ_i: polynomials of quadratic forms $\mathbf{x}_i^* G^{[i]}\mathbf{y}_i$, with $\mathbf{x}_i, \mathbf{y}_i = \mathbf{e}_i, \mathbf{v}_i$.

Then concentration of quadratic forms, *e.g.*

$$\Big|\mathbf{v}_i^* G^{[i]}\mathbf{v}_i - \mathbb{E}_i[\mathbf{v}_i^* G^{[i]}\mathbf{v}_i]\Big| \prec \frac{\|G^{[i]}\|_2}{N}\,, \qquad \mathbb{E}_i[\mathbf{v}_i^* G^{[i]}\mathbf{v}_i] = \mathrm{tr} G^{[i]}\,,$$

implies concentration of G_{ii}.

Green function entries

Aim:

$$G_{ii} \approx \frac{1}{a_i - \omega_B^c(z)}, \qquad \omega_B^c(z) = z - \frac{\mathrm{tr}\widetilde{B}G(z)}{\mathrm{tr}G(z)}, \qquad \widetilde{B} := UBU^*$$

From $(H - z)G(z) = 1$, we have $(a_i - z)G_{ii} = -(\widetilde{B}G)_{ii} + 1$, so that

$$G_{ii} = \frac{1}{a_i - z + \frac{(\widetilde{B}G)_{ii}}{G_{ii}}}.$$

We shall show:

Proposition.

For all $i = 1, 2, \ldots, N$,

$$(\widetilde{B}G)_{ii} \approx \frac{\mathrm{tr}\widetilde{B}G}{\mathrm{tr}G} G_{ii}.$$

Green function entries II

Proposition: $(\widetilde{B}G)_{ii} \approx \frac{\mathrm{tr}\widetilde{B}G}{\mathrm{tr}G} G_{ii}$.

Recall the decomposition $U = -\mathrm{e}^{\mathrm{i}\theta_i}(I - 2\mathbf{r}_i\mathbf{r}_i^*)U^{\langle i \rangle}$, where

$$\mathbf{r}_i := \frac{\mathbf{e}_i + \mathrm{e}^{-\mathrm{i}\theta_i}\mathbf{v}_i}{\|\mathbf{e}_i + \mathrm{e}^{-\mathrm{i}\theta_i}\mathbf{v}_i\|_2},$$

with $\mathbf{v}_i$ uniformly distributed on $\mathcal{S}_{\mathbb{C}}^{N-1}$. Set $\widetilde{B}^{\langle i \rangle} := U^{\langle i \rangle} B (U^{\langle i \rangle})^*$. Then,

$$\begin{aligned}(\widetilde{B}G)_{ii} &= \mathbf{e}_i^*(I - 2\mathbf{r}_i\mathbf{r}_i^*)\widetilde{B}^{\langle i \rangle}(I - 2\mathbf{r}_i\mathbf{r}_i^*)G\mathbf{e}_i \\ &\approx -\mathrm{e}^{\mathrm{i}\theta_i}\mathbf{v}_i^*\widetilde{B}^{\langle i \rangle}G\mathbf{e}_i .\end{aligned}$$

Main idea: Introduce two auxiliary quantities:

$$S_i(z) := \mathrm{e}^{\mathrm{i}\theta_i}\mathbf{v}_i^*\widetilde{B}^{\langle i \rangle}G(z)\mathbf{e}_i \approx -(\widetilde{B}G)_{ii}, \qquad T_i(z) := \mathrm{e}^{\mathrm{i}\theta_i}\mathbf{v}_i^* G(z)\mathbf{e}_i .$$

Derive a system of equations involving G_{ii}, $\mathbb{E}_i[S_i]$ and $\mathbb{E}_i[T_i]$ and solve $\mathbb{E}_i[S_i]$ from the system to get the proposition.

System of G, S and T

Computing $\mathbb{E}_i[S_i]$ and $\mathbb{E}_i[T_i]$ (using Gaussian approximation or Stein lemma), we get

$$\mathbb{E}_i[S_i] \approx \mathrm{tr}(\widetilde{B}G)\Big(\mathbb{E}_i[S_i] - b_i\mathbb{E}_i[T_i]\Big) + \mathrm{tr}(\widetilde{B}G\widetilde{B})\Big(G_{ii} + \mathbb{E}_i[T_i]\Big)\,,$$
$$\mathbb{E}_i[T_i] \approx \mathrm{tr}G\Big(\mathbb{E}_i[S_i] - b_i\mathbb{E}_i[T_i]\Big) + \mathrm{tr}(\widetilde{B}G)\Big(G_{ii} + \mathbb{E}_i[T_i]\Big)\,.$$

Solving the system for $\mathbb{E}_i[S_i]$ gives

$$\mathbb{E}_i[S_i] \approx -\frac{\mathrm{tr}(\widetilde{B}G)}{\mathrm{tr}G}G_{ii} + \left(\frac{\mathrm{tr}(\widetilde{B}G) - (\mathrm{tr}\widetilde{B}G)^2}{\mathrm{tr}G} + \mathrm{tr}(\widetilde{B}G\widetilde{B})\right)(G_{ii} + \mathbb{E}_i[T_i])\,.$$

Claim: The second term is negligible. ("Ward identity")

Proof: Averaging over i and using the facts $\mathbb{E}_i[S_i] \approx S_i \approx -(\widetilde{B}G)_{ii}$, and the less obvious fact $|\mathrm{tr}G - N^{-1}\sum_i \mathbb{E}_i[T_i]| \geq c$, which can be proved via a continuity argument.

Since $(\widetilde{B}G)_{ii} \approx \mathbb{E}_i[\widetilde{B}G)_{ii}] \approx -\mathbb{E}_i[S_i]$, we finally get

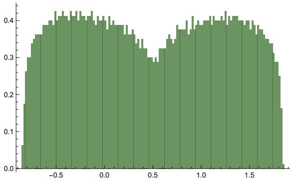
$$\left|(\widetilde{B}G)_{ii} - \frac{\mathrm{tr}(\widetilde{B}G)}{\mathrm{tr}G}G_{ii}\right| \prec \frac{1}{\sqrt{N\eta}}\,, \qquad z = E + \mathrm{i}\eta\,.$$

Ongoing work:

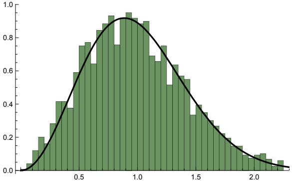
- Strong local law:

$$\left|m_H(z) - m_{\mu_A \boxplus \mu_B}(z)\right| \prec \frac{1}{N\eta}, \qquad \left|G_{ij}(z) - \delta_{ij}\frac{1}{a_i - \omega_B(z)}\right| \prec \frac{1}{\sqrt{N\eta}}.$$

- Derive the sine-kernel statistics of $H = A + UBU^*$ in the bulk.



Histogram of eigenvalues of H.



Histogram of eigenvalue gaps of H.

$$a_i \sim \mathrm{Bernoulli}(1/2), \quad b_i \sim \mathrm{Unif}(-1,1), \quad N = 3000$$

- Multiplicative model: $A^{1/2}UBU^*A^{1/2}$, global law (free multiplicative convolution) is known [Voiculescu '91].