Bi-free Infinite Divisibility

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classical infinite divisibility (de Finetti, Kolmogorov, Lévy, Hinčin, 1928-1937$^{(*)}$)

- X real random variable,
- $\varphi(t) = \mathrm{E}(e^{itX})$ characteristic function
- X is infinitely divisible if for all n there exist $X_1, \ldots, X_n$ independent and identically distributed such that

$$X \overset{\mathcal{D}}{\sim} X_1 + \cdots + X_n$$

- X is infinitely divisible $\Leftrightarrow \exists\ \alpha \in \mathbb{R}$ and σ pos. measure s.t.

$$\log(\varphi(t)) = \alpha t + \int \left(e^{itx} - 1 - \frac{itx}{1+x^2} \right) \frac{1+x^2}{x^2}\, d\sigma(t)$$

such an X is 'manifestly' infinite divisible

$^{(*)}$ *according to Steutel & van Harn, 2004*

free version, Bercovici-Voiculescu (1993)

Compactly supported case

- $R(z) = \kappa_1 + \kappa_2 z + \kappa_3 z^2 + \cdots$ is the R-transform of a compactly supported measure on $\mathbb{R}$

- X is freely infinitely divisible $\Leftrightarrow \exists$ and σ pos. measure s.t.

$$R(z) = \kappa_1 + \int \frac{z}{1 - tz} \, d\sigma(t)$$

$\Leftrightarrow$ R can be extended to $\mathbf{C}^+$ and maps $\mathbf{C}^+$ to $\mathbf{C}^+$

$\Leftrightarrow$ other equivalences . . .

such an X is 'manifestly' infinite divisible

Triangular Arrays (Nica-Speicher, 2006)

- suppose that for each N,
  $a_{N,1}, \ldots, a_{N,N} \in (\mathcal{A}_N, \varphi_N)$ free and identically dist.
- $a_{N,1} + \cdots + a_{N,N} \xrightarrow{\text{dist}} b \in (\mathcal{A}, \varphi)$

$\Leftrightarrow$ $\lim_N N\varphi_N(a_{N,1}^n)$ exists (and $= \kappa_n(b, \ldots, b)$ if limit exists)

(*use moment-cumulant formula and find leading order terms*)

this condition implies

- $\{\kappa_n\}_n$ (*cumulants of* b) are conditionally positive, which means

$$\sum_{m,n \geqslant 1} \alpha_m \overline{\alpha_n} \kappa_{m+n} = \lim_N N\varphi_N\Big(\big(\sum_m \alpha_m a^m \big) \big(\sum_n \alpha_n a^n \big)^* \Big) \geqslant 0$$

- which implies that $\exists$ a finite pos. measure σ s.t.

$$\kappa_{n+2} = \int t^n \, d\sigma(t)$$

conditionally positive sequences

- if the cumulants of b satisfy

$$\kappa_{n+2} = \int t^n \, d\sigma(t)$$

- then the R-transform of b can be written

$$R(z) = \sum_{n \geqslant 0} \kappa_{n+1} z^n = \kappa_1 + z \sum_{n \geqslant 0} \kappa_{n+2} z^n = \kappa_1 + \int \sum_{n \geqslant 0} (tz)^n \, d\sigma(t)$$

$$= \kappa_1 + \int \frac{z}{1 - tz} \, d\sigma(t)$$

- also such a $\{\kappa_n\}_n$ produces an inner product on $\mathbf{C}_0[X]$ = poly. variable X without constant term, let $\mathcal{H}$ be the corresponding Hilbert space and $\mathcal{F}(\mathcal{H})$ the full Fock space over $\mathcal{H}$

Fock space

- $\mathcal{H}$ a Hilbert space,
- $\xi \in \mathcal{H}$, $\ell(\xi)$ = left creation operator and
- $\ell(\xi)^*$ = left annihilation operator
- $T \in B(\mathcal{H})$, $\Lambda(T)\Omega = 0$,
  $\Lambda(T)(\xi_1 \otimes \cdots \otimes \xi_n) = T(\xi_1) \otimes \cdots \otimes \xi_n$
- for $Y_1 = \ell(\xi)$, $Y_2 = \ell(\eta)^*$, $Y_3 = \Lambda(T_*)$, $Y_4 = \alpha I$ then the only non-vanishing cumulant of $\kappa_n(Y_{i_1}, \ldots, Y_{i_n})$ is $\kappa_n(\ell(\eta)^*, \Lambda(T_1), \ldots, \Lambda(T_{n-2}), \ell(\xi)) = \langle T_1 \cdots T_{n-1}\xi, \eta\rangle$
  (*use limit theorem from 2 pages back*)

circle closed

- if $\{t_n\}_n$ is conditionally positive and $\mathcal{H}$ the Hilbert space obtained from $\mathbf{C}_0[X]$ we let X be the operator of left multiplication on $\mathcal{H}$ (bounded because of growth assumptions on $\{t_n\}_n$), also we let

$$b = \ell(X) + \ell(X)^* + \Lambda(X) + t_1 \in B(\mathcal{F}(\mathcal{H}))$$

  then $\kappa_n(b, \ldots, b) = t_n$ so $\{t_n\}_n$ is the cumulant sequence of a bounded self-adjoint operator
- operators of the form $\ell(X) + \ell(X)^* + \Lambda(X) + t_1$ are 'manifestly' freely infinitely divisible$^{(*)}$:

$$\ell\Big(\frac{X \oplus 0 \oplus \cdots 0}{\sqrt{N}}\Big) + \ell\Big(\frac{X \oplus 0 \oplus \cdots 0}{\sqrt{N}}\Big)^* + \Lambda(X \oplus 0 \oplus \cdots 0) + \frac{t_1}{N}$$

$$\vdots$$

$$\ell\Big(\frac{0 \oplus 0 \oplus \cdots X}{\sqrt{N}}\Big) + \ell\Big(\frac{0 \oplus 0 \oplus \cdots X}{\sqrt{N}}\Big)^* + \Lambda(0 \oplus 0 \oplus \cdots X) + \frac{t_1}{N}$$

$^{(*)}$ because Hilbert space is infinitely divisible

bi-freeness (*slightly simplified*)

- $\mathcal{X}_i = \mathbf{C}\xi_i \oplus \mathring{\mathcal{X}}_i$ vector spaces with distinguished subspace of co-dimension 1
- $(\mathcal{X}, \mathring{\mathcal{X}}, \xi) = *_i(\mathcal{X}_i, \mathring{\mathcal{X}}_i, \xi_i) = \mathbf{C}\xi \oplus \sum_{n \geqslant 1} \sum_{i_1 \neq \cdots \neq i_n}^{\oplus} \mathring{\mathcal{X}}_{i_1} \otimes \cdots \otimes \mathring{\mathcal{X}}_{i_n}$
- $l_r, r_i : \mathcal{L}(\mathcal{X}_i) \longrightarrow \mathcal{L}(\mathcal{X})$ "left" and "right" actions
- $(\mathcal{A}_i, \mathcal{B}_i) \subset \mathcal{L}(\mathcal{X}_i)$, a pair of faces, act on $\mathcal{X}$ via l_i and r_i
- $\langle \cdot \xi, \xi \rangle$ gives a state on the pairs $(l_i(\mathcal{A}_i), r_i(\mathcal{B}_i))$
- the pairs of faces (algebras) are *bi-free* by construction
- $\exists$? a description of bi-freeness without explicit use of free products, a challenge no cumulantologist can resist

bi-free cumulants (Mastnak-Nica)

- given $\chi : [n] \to \{l, r\}$ let $\chi^{-1}(l) = \{i_1 < \cdots < i_p\}$ and $\chi^{-1}(r) = \{j_1 < \cdots < j_{n-p}\}$
- usual non-crossing partitions are with respect to the order $(1, 2, 3, \ldots, n)$
- $NC_\chi(n)$ are non-crossing with respect to $(i_1, \ldots, i_p, j_{n-p}, \ldots, j_i)$
- $$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC_\chi(n)} \kappa_\pi^\chi(a_1, \ldots, a_n)$$

  (*moment-cumulant formula*)
- bi-freeness $\Leftrightarrow$ vanishing of mixed bi-free cumulants

  (*Charlesworth, Nelson & Skoufranis*)

bi-variate case: $[a, b] = 0$

- suppose a and b are commuting self-adjoint operators in a C^*-algebra with a state φ
- get $\mu \in \mathcal{M}(\mathbb{R}^2)$ a compactly supported probability measure
- given $\chi : [n] \to \{l, r\}$ let $c_1, \ldots, c_n$ be defined by $c_i = a$ if $\chi_i = l$ and $c_i = b$ if $\chi_i = r$
- $\kappa_n^\chi(c_1, \ldots, c_n)$ only depends on $\#(\chi^{-1}(l))$ and $\#(\chi^{-1}(r))$
- $\kappa_{m,n}(a, b)$ means m occurrences of a and n occurrences of b

$$R_{a,b}(z, w) = \sum_{\substack{m,n \geqslant 0 \\ m+n \geqslant 1}} \kappa_{m,n} z^m w^n, \; G(z, w) = \varphi((z-a)^{-1}(w-b)^{-1})$$

- $R_{a,b}(z, w) = zR_a(z) + wR_b(w) + 1 - \dfrac{zw}{G(K_a(z), K_b(w))}$
- $\mu_1 \boxplus\boxplus \mu_2$ is the distribution of the pair $(a_1 + a_2, b_1 + b_2)$ where (a_1, b_1) and (a_2, b_2) are bi-free

bi-free infinite divisibility

- if for every N we can find μ_N such that $\mu = \mu_N^{\boxplus\boxplus N}$ then μ is *bi-freely infinitely divisible*

THM: T.F.A.E.

1. μ bi-freely infinitely divisible
2. $\{\kappa_{m,n}\}_{m,n}$ are conditionally positive and conditionally bounded 2-sequences (*to be explained*)
3. $R_{a,b}$ has the integral representation

$$R_{a,b}(z,w) = zR_1(z) + wR_2(w) + \int \frac{z}{1-zs}\frac{w}{1-wt}\,d\rho(s,t)$$

with $R_1(z) = \kappa_{1,0} + \int \frac{z}{1-zs}\,d\rho_1(s,t)$, $R_2(w) = \kappa_{0,1} + \int \frac{w}{1-wt}\,d\rho_2(s,t)$
ρ_1 and ρ_2 compactly supported, ρ a signed Borel measure with compact support and

$$|\rho(\{0,0\})|^2 \leqslant \rho_1(\{0,0\})\rho_2(\{0,0\}), t\,d\rho_1(s,t) = s\,d\rho(s,t), s\,d\rho_2(s,t) = t\,d\rho(s,t)$$

conditionally positive and cond. bounded

- $\mathbf{C}_0[x, y]$ polynomials in commuting variables without constant term
- $\langle x^{m_1} y^{n_1}, x^{m_2} y^{n_2} \rangle = \kappa_{m_1+m_2, n_1+n_2}$ is a positive semi-def. inner product (*conditionally positive*)
- $\exists\, L > 0$ s.t. $|\langle x^m y^n p, p \rangle| \leqslant L^{m+n} \langle p, p \rangle$ (*conditionally bounded*)
- inner product on $\mathbf{C}_0[x, y]$ gives Hilbert space $\mathcal{H}$ and two multiplication operators T_1 (by x) and T_2 (by y) with spectral measures E_1 and E_2 (note $T_1(y) = T_2(x)$)
- $\rho([c_1, d_1] \times [c_2, d_2]) := \langle E_1([c_1, d_1])x, E_2([c_2, d_2])y \rangle$
- $\displaystyle\sum_{m,n \geqslant 1} \kappa_{m,n} z^m w^n \overset{(*)}{=} \int \frac{z}{1 - zs} \frac{w}{1 - wt} \, d\rho(s, t)$ (*by calculation*)
- $\theta^{(1)}_{m,n} = \kappa_{m+2,n}$, $\theta^{(2)}_{m,n} = \kappa_{m,n+2}$ give positive finite compactly supported measures ρ_1 and ρ_2
- $\int (s^m t^n) t \, d\rho_1(s, t) = \kappa_{m+2,n+1} = \int (s^m t^n) s \, d\rho(s, t)$ (by $(*)$)

bi-partite infinitely divisible operators

- $\mathcal{H}$ a Hilbert space, $\mathcal{F}(\mathcal{H})$ the full Fock space over $\mathcal{H}$
- $f, g \in \mathcal{H}$, $T_1 = T_1^*, T_2 = T_2^* \in B(\mathcal{H})$
- $a = \ell(f) + \ell(f)^* + \Lambda_l(T_1) + \lambda_1 \in B(\mathcal{F}(\mathcal{H}))$
- $b = r(g) + r(g)^* + \Lambda_r(T_2) + \lambda_2 \in B(\mathcal{F}(\mathcal{H}))$
- a, b commute iff $[T_1, T_2] = 0$, $T_1(g) = T_2(f)$, $\langle f, g\rangle \in \mathbb{R}$
- $a_{N,1} = \ell\Big(\frac{f \oplus 0 \oplus \cdots \oplus 0}{\sqrt{N}}\Big) + \ell\Big(\frac{f \oplus 0 \oplus \cdots \oplus 0}{\sqrt{N}}\Big)^* + \Lambda_l(T_1 \oplus 0 \oplus \cdots \oplus 0) + \frac{\lambda_1}{N}$
- $b_{N,1} = r\Big(\frac{g \oplus 0 \oplus \cdots \oplus 0}{\sqrt{N}}\Big) + r\Big(\frac{g \oplus 0 \oplus \cdots \oplus 0}{\sqrt{N}}\Big)^* + \Lambda_r(T_2 \oplus 0 \oplus \cdots \oplus 0) + \frac{\lambda_2}{N}$
- (a, b) bi-freely infinite divisible
- $\kappa_{m,n}(a, b) = \langle T_1^{m-1} f, T_2^{n-1} g\rangle$, $\kappa_{m,0} = \langle T_1^{m-2} f, f\rangle$, $\kappa_{1,0} = \lambda_1$

example: bi-free Poisson

- $(\alpha, \beta) \in \mathbb{R}^2, \lambda > 0$
- $\mu_N = \left(1 - \frac{\lambda}{N}\right)\delta_{(0,0)} + \frac{\lambda}{N}\delta_{(\alpha,\beta)}$
- $\mu = \lim_N \mu_N^{\boxplus\boxplus N}$ is bi-freely infinite divisible
- has bi-free cumulants $\kappa_{m,n} = \lambda\alpha^m\beta^n$ (*use limit theorem*)
- and $R(z,w) = \sum_{\substack{m,n \geqslant 0 \\ m+n \geqslant 1}} \kappa_{m,n} z^m w_n = \sum_{\substack{m,n \geqslant 0 \\ m+n \geqslant 1}} \lambda(\alpha z)^m(\beta w)^n$

$= \lambda z\left(\alpha + \frac{\alpha^2 z}{1-\alpha z}\right) + \lambda w\left(\beta + \frac{\beta^2 w}{1-\beta w}\right) + \frac{\lambda \alpha z \beta w}{(1-\alpha z)(1-\beta w)}$

$\rho_1(s,t) = \lambda s^2 \delta_{(\alpha,\beta)}$, $\rho_2(s,t) = \lambda t^2 \delta_{(\alpha,\beta)}$, $\rho(s,t) = \lambda st\delta_{(\alpha,\beta)}$
(ρ positive when $\alpha\beta > 0$)