On noncommutative distributional symmetries and de Finetti theorems associated with them

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Definitions

- Probability space $(\mathcal{A}, \phi)$
  $\mathcal{A}$ is a von Neumann algebra.
  ϕ is a normal state not necessarily faithful, but the GNS representation associated with ϕ is faithful.
  $x \in \mathcal{A}$ random variable.
- Joint distribution of $\{x_i | i \in I\}$, $\mu : \mathbb{C}\langle X_i | i \in I\rangle \to \mathbb{C}$ defined by

$$\mu(X_{i_1}^{k_1} X_{i_2}^{k_2} \cdots X_{i_n}^{k_n}) = \phi(x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_n}^{k_n}),$$

- An operator valued probability space $(\mathcal{A}, \mathcal{B}, E : \mathcal{A} \to \mathcal{B})$ consists of an algebra $\mathcal{A}$, a subalgebra $\mathcal{B}$ of $\mathcal{A}$ and a $\mathcal{B} - \mathcal{B}$ bimodule linear map $E : \mathcal{A} \to \mathcal{B}$, i.e.

$$E[b_1 a b_2] = b_1 E[a] b_2, \ \ E[b] = b$$

  for all $b_1, b_2, b \in \mathcal{B}$ and $a \in \mathcal{A}$.

Definitions

Definition

For an algebra $\mathcal{B}$, $\mathcal{B}\langle X\rangle$ is freely generated by $\mathcal{B}$ and the indeterminant X and $\mathcal{B}\langle X\rangle_0$ is a subalgebra of $\mathcal{B}\langle X\rangle$ which does not contain a constant term in $\mathcal{B}$.

Definition

$\{x_i\}_{i\in I} \subset (\mathcal{A}, \mathcal{B}, E : \mathcal{A} \to \mathcal{B})$ is said to be conditional independent over $\mathcal{B}$ if

$$E[p_1(x_{i_1})p_2(x_{i_2})\cdots p_n(x_{i_n})] = E[p_1(x_{i_1})]E[p_2(x_{i_2})]\cdots E[p_n(x_{i_n})]$$

whenever $i_1, \cdots, i_n$ are pairwise different and $p_1, \cdots, p_n \in \mathcal{B}\langle X\rangle$.

Classical symmetries

A finite sequence of random variables $(\xi_1, \xi_2, ..., \xi_n)$ is said to be exchangeable if

$$(\xi_1, ..., \xi_n) \stackrel{d}{=} (\xi_{\sigma(1)}, ..., \xi_{\sigma(n)}), \quad \forall \sigma \in S_n,$$

where S_n is the permutation group of n elements.
Compare with exchangeability, there is a weaker condition of spreadability: $(\xi_1, ..., \xi_n)$ is said to be spreadable if for any $k < n$, we have

$$(\xi_1, ..., \xi_k) \stackrel{d}{=} (\xi_{l_1}, ..., \xi_{l_k}), \quad 1 \leq l_1 < l_2 < \cdots < l_k \leq n$$

Note that i.i.d $\Rightarrow$ conditionally i.i.d $\Rightarrow$ exchangeability $\Rightarrow$ spreadability.

de Finetti Theorem

For infinite sequences of commutative random variables, we have

Theorem (de Finetti 1930s)

Infinite sequences of exchangeable random variables are conditionally i.i.d.

Theorem (Ryll-Nardzewski 1957)

Infinite sequences of spreadable random variables are conditionally i.i.d.

Therefore, Conditionally i.i.d $\Longleftrightarrow$ exchangeability $\Longleftrightarrow$ spreadability.

Köstler Theorem

In noncommutative probability, for infinite sequences, spreadability$\not\Rightarrow$ exchangeability $\not\Rightarrow$ any independence relation.

Definitions

Definition

$(\mathcal{A}, \mathcal{B}, E : \mathcal{A} \to \mathcal{B})$ such that $\mathcal{A}$ and $\mathcal{B}$ are unital. A family of $(x_i)_{i \in I}$ is said to be freely independent over $\mathcal{B}$, if

$$E[p_1(x_{i_1})p_2(x_{i_2}) \cdots p_n(x_{i_n})] = 0,$$

whenever $i_1 \neq i_2 \neq \cdots \neq i_n$, $p_1, ..., p_n \in \mathcal{B}\langle X \rangle$ and $E[p_k(x_{i_k})] = 0$ for all k.

Definition

$\{x_i\}_{i \in I} \subset (\mathcal{A}, \mathcal{B}, E : \mathcal{A} \to \mathcal{B})$ is said to be Boolean independent over $\mathcal{B}$ if

$$E[p_1(x_{i_1})p_2(x_{i_2}) \cdots p_n(x_{i_n})] = E[p_1(x_{i_1})]E[p_2(x_{i_2})] \cdots E[p_n(x_{i_n})]$$

whenever $i_1, \cdots, i_n \in I$, $i_1 \neq i_2 \neq \cdots \neq i_n$ and $p_1, \cdots, p_n \in \mathcal{B}\langle X \rangle_0$.

Definition

$\{x_i\}_{i\in I}$ is said to be monotonically independent over $\mathcal{B}$ if

$$\begin{aligned} & E[p_1(x_{i_1})\cdots p_{k-1}(x_{i_{k-1}})p_k(x_{i_k})p_{k+1}(x_{i_{k+1}})\cdots p_n(x_{i_n})] \\ = \; & E[p_1(x_{i_1})\cdots p_{k-1}(x_{i_{k-1}})E[p_k(x_{i_k})]p_{k+1}(x_{i_{k+1}})\cdots p_n(x_{i_n})] \end{aligned}$$

whenever $i_1,\cdots,i_n \in I$, $i_1 \neq i_2 \neq \cdots \neq i_n$, $i_{k-1} < i_k > i_{k+1}$ and $p_1,\cdots,p_n \in \mathcal{B}\langle X\rangle_0$.

Quantum symmetries

Definition

$A_s(n)$ is the universal unital C^*-algebra generated by $(u_{i,j})_{i,j=1,\cdots n}$:

- $u_{i,j}^* = u_{i,j} = u_{i,j}^2$ for all $i, j = 1, \cdots, n$.
- For each $i = 1, \cdots, n$ and $k \neq l$ we have

$$u_{ik}u_{il} = 0 \quad \text{and} \quad u_{ki}u_{li} = 0; \quad .$$

- for each $i = 1, \cdots, n$ we have

$$\sum_{k=1}^{n} u_{ik} = 1 = \sum_{k=1}^{n} u_{ki}.$$

$A_s(n)$ is a compact quantum group in sense of Woronowicz.

Quantum symmetries

- Right coaction of $A_s(n)$ on $\mathbb{C}\langle X_1, ..., X_n\rangle$ is a unital homomorphism $\alpha_n : \mathbb{C}\langle X_1, ..., X_n\rangle \to \mathbb{C}\langle X_1, ..., X_n\rangle \otimes A_s(n)$ defined by

$$\alpha_n(X_i) = \sum_{j=1}^{n} X_j \otimes u_{j,i}$$

- $(x_1, ..., x_n) \subset \mathcal{A}$ is said to be quantum exchangeable if

$$\mu_{x_1,...,x_n}(p)1_{A_s(n)} = \mu_{x_1,...,x_n} \otimes id_{A_s(n)}(\alpha_n(p))$$

  for all $p \in \mathbb{C}\langle X_1, ..., X_n\rangle$.
- An infinite sequence $(x_i)_{i\in\mathbb{N}}$ is quantum exchangeable if all its finite subsequences are quantum exchangeable.

Free de Finetti Theorem

Definition

Let $(\mathcal{A}, \phi)$ be W^*-probability space with a faithful state, $\mathcal{A}$ is generated by $(x_i)_{i\in\mathbb{N}}$. The tail algebra of $(x_i)_{i\in\mathbb{N}}$ is

$$\mathcal{A}_{tail} = \bigcap_{n=1}^{\infty} vN\{x_k | k \geq n\},$$

where $vN\{x_k | k \geq n\}$ is the von Neumann algebra generated by $\{x_k | k \geq n\}$.

Free de Finetti Theorem

Theorem (Köstler 2010)

If $(x_i)_{i\in\mathbb{N}}$ are exchangeable, then $\exists$ a normal endomorphism $\alpha : \mathcal{A} \to \mathcal{A}$ such that $\alpha(x_i) = x_{i+1}$ for all $i \in \mathbb{N}$. Moreover,

$$E = WOT - \lim_{n\to\infty} \alpha^n$$

is a well defined conditional expectation from $\mathcal{A}$ onto $\mathcal{A}_{tail}$.

Free de Finetti Theorem

Theorem (Köstler & speicher 2009)

For infinite sequences, Quantum exchangeable $\Longleftrightarrow$ free with respect to $E : \mathcal{A} \to \mathcal{A}_{tail}$.

Boolean de Finetti Theorem

Definition

$B_s(n)$ is defined as the universal unital C^*-algebra generated by elements $u_{i,j}$ $(i, j = 1, \cdots n)$ and a projection $\mathbf{P}$ such that we have

- each $u_{i,j}$ is an orthogonal projection, i.e. $u_{i,j}^* = u_{i,j} = u_{i,j}^2$ for all $i, j = 1, \cdots, n$.
- $$u_{i,k}u_{i,l} = 0 \quad \text{and} \quad u_{k,i}u_{l,i} = 0$$
  whenever $k \neq l$.
- For all $1 \leq i \leq n$, $\mathbf{P} = \sum_{k=1}^{n} u_{k,i}\mathbf{P}$.

Boolean de Finetti Theorem

- Right coaction of $B_s(n)$ on $\mathbb{C}\langle X_1, ..., X_n\rangle$ is a unital homomorphism $\alpha_n : \mathbb{C}\langle X_1, ..., X_n\rangle \to \mathbb{C}\langle X_1, ..., X_n\rangle \otimes B_s(n)$ defined by

$$\alpha_n(X_i) = \sum_{j=1}^{n} X_j \otimes u_{j,i}$$

- $(x_1, ..., x_n) \subset \mathcal{A}$ is said to be Boolean exchangeable if

$$\mu_{x_1,...,x_n}(p)\mathbf{P} = \mathbf{P}\mu_{x_1,...,x_n} \otimes id_{B_s(n)}(\alpha_n(p))\mathbf{P}$$

  for all $p \in \mathbb{C}\langle X_1, ..., X_n\rangle$.
- An infinite sequence $(x_i)_{i\in\mathbb{N}}$ is Boolean exchangeable if all its finite subsequences are Boolean exchangeable.

Boolean de Finetti Theorem

Remark

There is no pair of Boolean independent random variables in probability spaces with faithful states. Therefore, in our framework, we just require the GNS representation associated with the state to be faithful.

Tail algebra

The tail algebra $\mathcal{T}$ of $(x_i)_{i\in\mathbb{N}}$ is defined by the following formula:

$$\mathcal{T} = \bigcap_{n=1}^{\infty} W^*\{x_k | k \geq n\},$$

where $W^*\{x_k | k \geq n\}$ is the WOT closure of the non-unital algebra generated by $\{x_k | k \geq n\}$. We call $\mathcal{T}$ the non-unital tail algebra of $(x_i)_{i\in\mathbb{N}}$

Boolean de Finetti Theorem

Theorem

Let $(\mathcal{A}, \phi)$ be a W^*-probability space and $(x_i)_{i\in\mathbb{N}}$ be an infinite sequence of selfadjoint random variables which generate $\mathcal{A}$ as a von Neumann algebra. Then the following are equivalent:

a) *The joint distribution of $(x_i)_{i\in\mathbb{N}}$ is Boolean exchangeable.*

b) *The sequence $(x_i)_{i\in\mathbb{N}}$ is identically distributed and Boolean independent with respect to a ϕ-preserving conditional expectation E onto the tail algebra of the $(x_i)_{i\in\mathbb{N}}$.*

Spreadability

Rephrasing spreadability in words of quantum maps:

$I_{k,n}$ set of increasing sequences $\mathcal{I} = (1 \le i_1 < \cdots < i_k \le n)$. For $1 \le i \le n$, $1 \le j \le k$, define $f_{i,j} : I_{k,n} \to \mathbb{C}$ by:

$$f_{i,j}(\mathcal{I}) = \begin{cases} 1, & i_j = i \\ 0, & i_j \neq i \end{cases} .$$

$C(I_{n,k})$ generated by the functions $f_{i,j}$.
$\exists \alpha : \mathbb{C}[X_1, ..., X_k] \to \mathbb{C}[X_1, ..., X_n] \otimes C(I_{k,n})$ define by:

$$\alpha : X_j = \sum_{i=1}^{n} X_i \otimes f_{i,j}, \quad \alpha(1) = 1_{C(I_{k,n})}$$

Spreadability

For fixed $k < n$,

$$\mu_{x_1,\dots,x_k}(p)1_{C(I_{n,k})} = \mu_{x_1,\dots,x_n} \otimes id_{C(I_{n,k})}(\alpha(p))$$

for all $p \in \mathbb{C}[x_1, \dots, x_k]$.

$\iff$

$$(\xi_1, \dots, \xi_k) \stackrel{d}{=} (\xi_{l_1}, \dots, \xi_{l_k}), \quad 1 \leq l_1 < l_2 < \cdots < l_k \leq n$$

Curran's quantum increasing space

Definition

For $k, n \in \mathbb{N}$ with $k \leq n$, the quantum increasing space $A(n, k)$ is the universal unital C^*-algebra generated by elements $\{u_{i,j} | 1 \leq i \leq n, 1 \leq j \leq k\}$ such that

1. Each $u_{i,j}$ is an orthogonal projection: $u_{i,j} = u^*_{i,j} = u^2_{i,j}$ for all $i = 1, ..., n; j = 1, ..., k$.
2. Each column of the rectangular matrix $u = (u_{i,j})_{i=1,...,n;j=1,...,k}$ forms a partition of unity: for $1 \leq j \leq k$ we have $\sum_{i=1}^{n} u_{i,j} = 1$.
3. Increasing sequence condition: $u_{i,j} u_{i',j'} = 0$ if $j < j'$ and $i \geq i'$.

Curran's quantum spreadability

For any natural numbers $k < n$, $\exists$ unital $*$-homomorphism $\alpha_{n,k} : \mathbb{C}\langle X_1, ..., X_k\rangle \to \mathbb{C}\langle X_1, ..., X_n\rangle \otimes A_i(n,k)$ such that:

$$\alpha_{n,k}(X_j) = \sum_{i=1}^{n} X_i \otimes u_{i,j}.$$

Definition

$(x_i)_{i=1,...,n}$ $A_i(n,k)$-spreadable if

$$\mu_{x_1,...,x_n}(p)1_{A_i(n,k)} = \mu \otimes id_{A_i(n,k)}(\alpha_{n,k}(p)),$$

for all $p \in \mathbb{C}\langle X_1, ..., X_k\rangle$. $(x_i)_{i=1,...,n}$ is said to be quantum spreadable if $(x_i)_{i=1,...,n}$ is $A_i(n,k)$-spreadable for all $k = 1, ..., n-1$.

Free extended de Finetti theorem

Theorem (Curran 2010)

In W^*-probability space $(\mathcal{A}, \phi)$, where ϕ is a faithful tracial state. For infinite sequences, quantum spreadable $\Longleftrightarrow$ free with respect to $E : \mathcal{A} \to \mathcal{A}_{tail}$ $\Longleftrightarrow$ quantum exchangeable.

Boolean spreadability

Inspired by $B_s(n)$, we can construct Boolean space of increasing spaces $B_i(n,k)$:

Definition

For $k, n \in \mathbb{N}$ with $k \leq n$, $B_i(k,n)$ is the unital universal C^*-algebra generated by elements $\{u_{i,j}^{(b)} | 1 \leq i \leq n, 1 \leq j \leq k\}$ and an invariant projection $\mathbf{P}$ such that

1. Each $u_{i,j}^{(b)}$ is an orthogonal projection: $u_{i,j}^{(b)} = (u_{i,j}^{(b)})^* = (u_{i,j}^{(b)})^2$ for all $i = 1, ..., n; j = 1, ..., k$.
2. For $1 \leq j \leq k$ we have $\sum_{i=1}^{n} u_{i,j}^{(b)} \mathbf{P} = \mathbf{P}$.
3. Increasing sequence condition: $u_{i,j}^{(b)} u_{i',j'}^{(b)} = 0$ if $j < j'$ and $i \geq i'$.

$\exists$ unital homomorphism $\alpha_{n,k}^{(b)} : \mathbb{C}\langle X_1, ..., X_k \rangle \to \mathbb{C}\langle X_1, ..., X_n \rangle \otimes B_i(n,k)$ determined by:

$$\alpha_{n,k}^{(b)}(x_j) = \sum^{n} x_i \otimes u_{i,j}^{(b)}$$

Boolean spreadability

Definition

$(x_i)_{i=1,...,n}$ in $(\mathcal{A}, \phi)$ is $B_i(n,k)$-spreadable if

$$\mu_{x_1,...,x_k}(p)\mathbf{P} = \mathbf{P}\mu_{x_1,...,x_n} \otimes id_{B_i(n,k)}(\alpha_{n,k}^{(b)}(p))\mathbf{P},$$

for all $p \in \mathbb{C}\langle X_1, ..., X_k\rangle$. $(x_i)_{i=1,...,n}$ is Boolean spreadable if $(x_i)_{i=1,...,n}$ is $B_i(n,k)$-spreadable for all $k = 1, ..., n-1$.

Monotone spreadability

Definition

For fixed $n, k \in \mathbb{N}$ and $k < n$, a monotone increasing sequence space $M_i(n,k)$ is the universal unital C^*-algebra generated by elements $\{u_{i,j}^{(m)}\}_{i=1,\dots,n; j=1,\dots,k}$

1. Each $u_{i,j}$ is an orthogonal projection;
2. Monotone condition: Let $P_j = \sum_{i=1}^{n} u_{i,j}^{(m)}$, $P_j u_{i'j'}^{(m)} = u_{i'j'}$ if $j' \leq j$.
3. $\sum_{i=1}^{n} u_{i,j}^{(m)} P_1 = P_1$ for all $1 \leq j \leq k$.
4. Increasing condition: $u_{i,j}^{(m)} u_{i',j'}^{(m)} = 0$ if $j < j'$ and $i \geq i'$.

We see that P_1 plays the role as the invariant projection $\mathbf{P}$ in the Boolean case. For consistency, we denote P_1 by $\mathbf{P}$.

Monotone spreadability

$\exists$ unital $*$- homomorphism $\alpha_{n,k}^{(m)} : \mathbb{C}\langle X_1, ..., X_k\rangle \to \mathbb{C}\langle X_1, ..., X_n\rangle \otimes M_i(n,k)$ such that

$$\alpha_{n,k}^{(m)}(X_j) = \sum_{i=1}^{n} X_i \otimes u_{i,j}^{(m)}.$$

Definition

A finite ordered sequence of random variables $(x_i)_{i=1,...,n}$ in $(\mathcal{A}, \phi)$ is said to be $M_i(n,k)$-invariant if their joint distribution $\mu_{x_1,...,x_n}$ satisfies:

$$\mu_{x_1,...,x_k}(p)\mathbf{P} = \mathbf{P}\mu_{x_1,...,x_n} \otimes id_{M_i(n,k)}(\alpha_{n,k}^{(m)}(p))\mathbf{P},$$

for all $p \in \mathbb{C}\langle X_1, ..., X_k\rangle$. $(x_i)_{i=1,...,n}$ is said to be monotonically spreadable if it is $M_i(n,k)$-invariant for all $k = 1, ..., n-1$.

Booolean exchangeability ⟶ *Boolean spreadability*

Boolean spreadability ⟶ *Monotone spreadability*

Monotone spreadability ⟶ *Quantum spreadability*

Booolean exchangeability ⟶ *Quantum exchangeability*

Quantum exchangeability ⟶ *Quantum spreadability*.

An unbounded spreadable sequence

Let $\mathcal{H}$ be the standard 2-dimensional Hilbert space with orthonormal basis

$$\{v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}.$$

Let $p, A, x \in B(\mathcal{H})$ be operators on $\mathcal{H}$ with the following matrix forms:

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $H = \bigotimes_{n=1}^{\infty} \mathcal{H}$ the infinite tensor product of $\mathcal{H}$. Let $\{x_i\}_{i=1}^{\infty}$ be a sequence of selfadjoint operators in $B(H)$ defined as follows:

$$x_i = \bigotimes_{n=1}^{i-1} A \otimes x \otimes \bigotimes_{m=1}^{\infty} p$$

Let ϕ be the vector state $\langle \cdot v, v \rangle$ on $\mathcal{H}$ and $\Phi = \bigotimes_{n=1}^{\infty} \phi$ be a state on $B(H)$.

An unbounded spreadable sequence

- $(x_i)_{i\in\mathbb{N}}$ is monotonically spreadable with respect to Φ and $\sup_i \|x_i\| = \infty$.
- Unilateral shift is unbounded.

To construct a conditional expectation, we need to consider bilateral sequences of random variables.

Tail algebra

Definition

Let $(\mathcal{A}, \phi)$ be a non-degenerated noncommutative W^*-probability space, $(x_i)_{i\in\mathbb{Z}}$ be a bilateral sequence of bounded random variables in $\mathcal{A}$ such that $\mathcal{A}$ is the WOT closure of the non-unital algebra generated by $(x_i)_{i\in\mathbb{Z}}$. The positive tail algebra $\mathcal{A}_{tail}^{+}$ of $(x_i)_{i\in\mathbb{Z}}$ is defined as following:

$$\mathcal{A}_{tail}^{+} = \bigcap_{k>0} \mathcal{A}_k^{+}.$$

where $\mathcal{A}_k^{+}$ is the WOT-closure of the non-unital algebra generated by $(x_i)_{i\geq k}$, In the opposite direction, we define the negative tail algebra $\mathcal{A}_{tail}^{-}$ of $(x_i)_{i\in\mathbb{Z}}$ as following:

$$\mathcal{A}_{tail}^{-} = \bigcap_{k<0} \mathcal{A}_k^{-}.$$

Tail algebra

If $(x_i)_{i\in\mathbb{Z}}$ is spreadable, then

- $\exists$ a normal automorphism $\alpha : \mathcal{A} \to \mathcal{A}$ such that $\alpha(x_i) = x_{i+1}$ for all $i \in \mathbb{Z}$.
- For $k \in \mathbb{Z}$, let $\mathcal{A}_k^+$ be the WOT-closure of the non-unital algebra generated by $(x_i)_{i\geq k}$, then

$$E^+ = \lim_{n\to\infty} \alpha^n$$

  defines a normal conditional expectation from $\mathcal{A}_k^+$ onto $\mathcal{A}_{tail}^+$.
- In general, E^+ can not be extended to the whole algebra $\mathcal{A}$.
- In the similar way, we can define conditional expectation E^-.

Monotone extended de Finetti theorem

Theorem

Let $(\mathcal{A}, \phi)$ be a non-degenerated W^*-probability space and $(x_i)_{i\in\mathbb{Z}}$ be a bilateral infinite sequence of selfadjoint random variables which generate $\mathcal{A}$. Let $\mathcal{A}_k^+$ be the WOT closure of the non-unital algebra generated by $\{x_i | i \geq k\}$. Then the following are equivalent:

a) *The joint distribution of $(x_i)_{i\in\mathbb{Z}}$ is monotonically spreadable.*

b) *For all $k \in \mathbb{Z}$, there exits a ϕ-preserving conditional expectation $E_k : \mathcal{A}_k^+ \to \mathcal{A}_{tail}^+$ such that the sequence $(x_i)_{i\geq k}$ is identically distributed and monotone with respect E_k. Moreover, $E_k|_{\mathcal{A}_{k'}} = E_{k'}$ when $k \geq k'$.*

Monotone extended de Finetti theorem

Proposition

Let $(\mathcal{A}, \phi)$ be a non-degenerated W^*-probability space and $(x_i)_{i\in\mathbb{Z}}$ be a bilateral infinite sequence of selfadjoint random variables which generate $\mathcal{A}$. If $(x_i)_{i\in\mathbb{Z}}$ is monotonically spreadable, then the negative conditional expectation E^- can be extended to the whole algebra $\mathcal{A}$.

Boolean extended de Finetti theorem

Lemma

$(\mathcal{A}, \phi)$ is a W^*-probability space with a non-degenerated normal state and $\mathcal{A}$ is generated by a bilateral sequence of random variables $(x_i)_{i\in\mathbb{Z}}$ and $(x_i)_{i\in\mathbb{Z}}$ are Boolean spreadable. Then, E^- and E^+ can be extended to the whole algebra $\mathcal{A}$. Moreover, $E^- = E^+$

Boolean extended de Finetti theorem

Theorem

Let $(\mathcal{A}, \phi)$ be a non degenerated W^*-probability space and $(x_i)_{i\in\mathbb{Z}}$ be a bilateral infinite sequence of selfadjoint random variables which generate $\mathcal{A}$ as a von Neumann algebra. Then the following are equivalent:

a) *The joint distribution of $(x_i)_{i\in\mathbb{N}}$ is Boolean spreadable.*

b) *The sequence $(x_i)_{i\in\mathbb{Z}}$ is identically distributed and Boolean independent with respect to the $\phi-$preserving conditional expectation E^+ onto the non unital positive tail algebra of the $(x_i)_{i\in\mathbb{Z}}$*

Free and classical de Finetti theorems

In 2009, Banica and Speicher found some universal conditions which can define some new quantum groups which are called easy quantum groups. By using the invariance conditions associated with those easy quantum groups, Banica, Curran and Speicher found more de Finetti theorems for both classical independence and free independence.

Conditions for easy quantum groups

Let $u \in M_n(\mathcal{A})$ be a matrix over a C^*-algebra $\mathcal{A}$ the pair u is called:

- Orthogonal, if all entries of u are selfadjoint, and $uu^t = u^t u = 1_n$,
- magic, if it is orthogonal, and its entries are projections.
- cubic, if it is orthogonal, and $u_{i,j}u_{i,k} = u_{j,i}u_{j,k} = 0$, for $j \neq k$.
- bistochastic, if it is orthogonal, and $\sum_{j=1}^{n} u_{i,j} = \sum_{j=1}^{n} u_{j,i} = 1_n$, for $j \neq k$.
- magic',if it is cubic, with the same sum on rows and columns.
- bistochastic',if it is orthogonal, with the same sum on rows and columns

The universal quantum groups associated with these four conditions are $A_o(n)$, $A_s(n)$, $A_h(n)$, $A_b(n)$, $A_{s'}(n)$ and $A_{b'}(n)$.

Universal conditions for Boolean independence

Let $u \in M_n(\mathcal{A})$ be a matrix over a C^*-algebra $\mathcal{A}$ and $\mathbf{P}$ be a projection in $\mathcal{A}$, the pair $(u, \mathbf{P})$ is called:

- **P**-orthogonal, if all entries of u are selfadjoint, and $uu^t\mathbf{P} = u^t u\mathbf{P} = 1_n \otimes \mathbf{P}$,
- **P**-magic, if it is $\mathbf{P}-$orthogonal, and its entries are projections.
- **P**-cubic, if it is $\mathbf{P}-$orthogonal, and $u_{i,j}u_{i,k} = u_{j,i}u_{j,k} = 0$, for $j \neq k$.
- **P**-bistochastic, if it is $\mathbf{P}-$orthogonal, and $\sum_{j=1}^{n} u_{i,j}\mathbf{P} = \sum_{j=1}^{n} u_{j,i}\mathbf{P} = \mathbf{P}$, for $j \neq k$.
- **P**-magic', if it is $\mathbf{P}-$cubic, with the same sum on rows and columns.
- **P**-bistochastic',if it is $\mathbf{P}-$orthogonal, with the same sum on rows and columns.

Then, we can define quantum semigroup associated with these four conditions, which are $B_o(n)$, $B_s(n)$, $B_h(n)$, $B_b(n)$, $B_{s'}(n)$ and $B_{b'}(n)$.

Suppose ϕ is faithful. Let $\{E(n)\}_{n\in\mathbb{N}}$ be a sequence of orthogonal Hopf algebras such that $A_s(n) \subseteq E(n) \subseteq A_o(n)$ for each n. If the joint distribution of $(x_i)_{i\in\mathbb{N}}$ is $E(n)$ invariant, then there are a W^*-subalgebra $1 \subseteq \mathcal{B} \subseteq \mathcal{A}$ and a ϕ-preserving conditional expectation $E : \mathcal{A} \to \mathcal{B}$ such that

Theorem

1. *If* $E(n) = A_s(n)$ *for all* n*, then* $(x_i)_{i\in\mathbb{N}}$ *are freely independent and identically distributed with respect to* E*.*
2. *If* $A_s(n) \subseteq E(n) \subseteq A_h(n)$ *for all* n *and there exists a* k *such that* $E(k) \neq A_s(k)$*, then* $(x_i)_{i\in\mathbb{N}}$ *are freely independent and have identically symmetric distribution with respect to* E*.*
3. *If* $A_s(n) \subseteq E(n) \subseteq A_b(n)$ *for all* n *and there exists a* k *such that* $E(k) \neq A_s(k)$*, then* $(x_i)_{i\in\mathbb{N}}$ *are conditionally independent and have identically shifted-semicircular distribution with respect to* E*.*
4. *If there exist* k_1, k_2 *such that* $E(k_1) \not\subseteq A_h(k_1)$ *and* $E(k_2) \not\subseteq A_b(k_2)$*, then* $(x_i)_{i\in\mathbb{N}}$ *are freely independent and have centered semicircular distribution with respect to* E*.*

Suitable Framework

Remark

If the framework is too large, we would not get de Finetti theorem for certain independence. If the framework is too small, we would get trivial result i.e. all random variables are identical too each others.

Thank You!