De Finetti theorems for a Boolean analogue of easy quantum groups

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Free Probability and the Large N limit, V

The University of California, Berkeley

Free and Boolean de Finetti theorems

Free and Boolean de Finetti theorems:

1. Free de Finetti theorem for A_s (C. Köstler and R. Speicher, 2009)
2. Free de Finetti theorems for free quantum groups (T. Banica, S. Curran and R. Speicher, 2012)
3. Boolean de Finetti theorem for $\mathcal{B}_s$ (W.Liu, 2015)

Our result: Find general Boolean de Finetti theorem for a Boolean analogue of free quantum groups.
Our strategy: Find a nice class of interval partitions and use BCS's framework.

Liu himself proved Boolean de Finetti theorems for quantum semigroups by a different way.

De Finetti theorems for free quantum groups

(M, φ) : v.N.alg and faithful normal state
$x_n \in M_{s.a.}$ $(n \in \mathbb{N})$

Invariant under	iff	$(x_n)_{n\in\mathbb{N}}$ is
S_n^+		free i.i.d. over tail (*)
O_n^+		(*) & centered semicircular
B_n^+		(*) & semicircular
H_n		(*) & even

Symmetries	Categories of partitions	Distributions
S_n^+	NC	free i.i.d. over tail (*)
O_n^+	NC_2	(*) & centered semicircular
B_n^+	NC_b	(*) & semicircular
H_n	NC_h	(*) & even

Tannaka-Klein duality : A sequence of free quantum groups
$(A_x(n))_{n\in\mathbb{N}} \overset{1:1}{\Longleftrightarrow}$ A category of noncrossing partitions NC_x
Cumulants-Moments formula

Review on conditional Boolean independence

Definition

$\eta \colon N \hookrightarrow M$: a normal embedding of v.N. algebras w/ $\eta(1_N) \neq 1_M$,
$E \colon M \to N$: a normal conditional expecation w/ $E \circ \eta = id_N$.

$(x_j \in M_{s.a.})_{j \in J}$ is *Boolean independent w.r.t.* E if

$$E[f_1(x_{j_1}) f_2(x_{j_2}) \cdots f_k(x_{j_k})] = E[f_1(x_{j_1})] E[f_2(x_{j_2})] \cdots E[f_k(x_{j_k})],$$

whenever $j_1 \neq j_2 \neq \cdots \neq j_k$ and

$$f_1, \ldots, f_k \in N\langle X \rangle^{\circ}.$$

$$(i.e.\ N - polynomials\ without\ constant\ terms)$$

Liu's Boolean de Finetti theorem

Liu defined a quantum semigroup $\mathcal{B}_s(n)$ as the universal unital C*-algebra generated by projections $P, U_{i,j} (i, j = 1, \ldots, n)$ and relations such that

$$\sum_{j=1}^{n} U_{ij} P = P, \sum_{i=1}^{n} U_{ij} P = P,$$

$$U_{i_1 j} U_{i_2 j} = 0, \text{ if } i_1 \neq i_2, U_{i j_1} U_{i j_2} = 0, \text{ if } j_1 \neq j_2.$$

Theorem (Liu, 2015)

(M, φ) : a v.N.algebra & a nondegenerate normal state.
$x_j \in M_{s.a.}$, $j \in \mathbb{N}$ with $M = W^*(ev_x(\mathscr{P}_\infty^o))$ where
$\mathscr{P}_\infty^o := \{f \in \mathbb{C}\langle (X_j)_{j\in\mathbb{N}}\rangle \mid f(0) = 0\}$
TFAE.

1. *The joint distribution of $(x_j)_{j\in\mathbb{N}}$ is invariant under the coaction of B_s.*
2. *There exists a normal conditional expectation $E_{tail} : M \to M_{\mathrm{tail}} := \bigcap_{n=1}^{\infty} \overline{\mathrm{ev}_x(\mathscr{P}_{\geq n}^o)}^{\sigma w}$ and $(x_j)_{j\in\mathbb{N}}$ is Boolean i.i.d. over tail.*

Our strategy

Aim : Fill the missing piece in Boolean de Finetti theorem.

Our strategy : Find a nice class of interval partitions and use BCS's framework.

Difficulty: Bad-behavors of non-unital embeddings and non-faithful states

Review on category of partitions

$P(k,l)$: the set of all partitions of the disjoint union $[k] \amalg [l]$, where $[k] = \{1, 2, \ldots, k\}$ for $k \in \mathbb{N}$.
Such a partition will be pictured as

$$p = \left\{ \begin{matrix} 1 \ldots k \\ \mathcal{P} \\ 1 \ldots l \end{matrix} \right\}$$

where $\mathcal{P}$ is a diagram joining the elements in the same block of the partition. Categorical operations:

$$\begin{aligned} p \otimes q &= \{\mathcal{P}\mathcal{Q}\} : \text{Horizontal concatenation} \\ pq &= \left\{ \begin{matrix} \mathcal{Q} \\ \mathcal{P} \end{matrix} \right\} - \{\text{closed blocks}\} : \text{Vertical concatenation} \\ p^* &= \{\mathcal{P}^{\curvearrowright}\} : \text{Upside-down turning} \end{aligned}$$

category of interval partitions

$NC := (NC(k,l))_{k,l}$: the family of all noncrossing partitions
$NC_x = \{NC_x(k,l)\}_{k,l}$, $NC_x(k,l) \subseteq NC(k,l)$ is *a category of noncrossing partitions* if

1. It is stable by categorical operations
2. $\sqcap \in NC_x(0,2)$
3. $| \in NC_x(1,1)$

$I(k) := \{\pi \in P(k) \mid \textit{interval partition}\}$, $I := (I(k) \times I(l))_{k,l}$

Definition (Category of interval partitions)

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2. $\sqcap \in I_x(0,2)$

Category of interval partitions

Remark

$$I_x(k,l) = I_x(k,0) \times I_x(0,l)$$

$$I_x(k) := I_x(0,k)$$

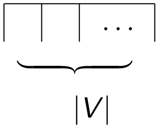
Example

The followings are categories of interval partitions.

1. $I_2 = (\{\pi \in I(k) \mid \textit{block size } 2\})_k$
2. $I_b = (\{\pi \in I(k) \mid \textit{block size} \leq 2\})_k$
3. $I_h = (\{\pi \in I(k) \mid \textit{block size even}\})_k$

Review on NC_x

To find the class of interval partitions suited to de Finetti, review on NC_x.
NC, NC_2, NC_b, and NC_h are **block-stable**,
i.e. for any $\pi \in NC_x$ and $V \in \pi$,



$$\in NC_x(|V|).$$

These four categories of noncrossing partitions are also **closed under taking an interval in NC,** i.e.

$$\rho, \sigma \in NC_x(k), \pi \in NC(k), \rho \leq \pi \leq \sigma \Longrightarrow \pi \in NC_x(k).$$

This condtition appears in Möbius inversions:

Review on Möbius function

Let $(Q, \leq)$ be a finite poset. The Möbius function
$\mu_Q \colon \{(\pi, \sigma) \in Q^2 \mid \pi \leq \sigma\} \to \mathbb{C}$
is defined by the following relations: for any $\pi, \sigma \in Q$ with $\pi \leq \sigma$,

$$\sum_{\substack{\rho \in Q \\ \pi \leq \rho \leq \sigma}} \mu_Q(\pi, \rho) = \delta(\pi, \sigma),$$

$$\sum_{\substack{\rho \in Q \\ \pi \leq \rho \leq \sigma}} \mu_Q(\rho, \sigma) = \delta(\pi, \sigma),$$

where if $\pi = \sigma$ then $\delta(\pi, \sigma) = 1$, otherwise, $\delta(\pi, \sigma) = 0$.

Closed under taking an interval

If $R \subseteq Q$ is closed under taking an interval in Q,

$$\mu_R(\pi, \sigma) = \mu_Q(\pi, \sigma).$$

Blockwise condition

We define a suitable class of interval partitions.

Definition (Blockwise condition)

Let D be a category of interval partition. D is said to be *blockwise* if

1. D is block-stable,
2. D is closed under taking an interval in I, i.e.,

$$\rho, \sigma \in D(k), \pi \in I(k), \rho \leq \pi \leq \sigma \Longrightarrow \pi \in D(k).$$

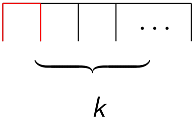
Key condition

If D is blockwise,

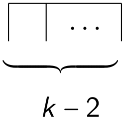
$$\mu_{D(k)}(\pi, \sigma) = \mu_{I(k)}(\pi, \sigma).$$

Pairing

By composition with the pair partition $\sqcap$ & **the unit partition** $|$,
it holds that

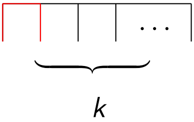


$\in NC_x(0,k) \Longrightarrow$

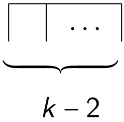


$\in NC_x(0,k-2)$.

I_x: a category of interval partitions
Becasue **the unit partition** $| \notin I_x(1,1)$, in general,



$\in I_x(0,k) \not\Longrightarrow$

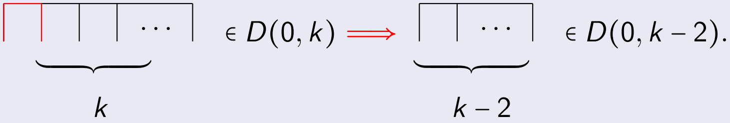


$\in I_x(0,k-2)$.

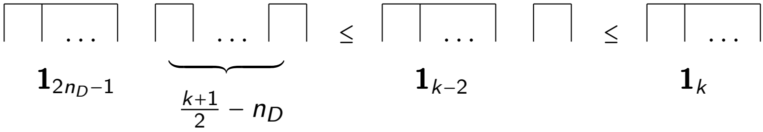
Pairing in blockwise category of interval partition

Lemma

D : a blockwise category of interval partitions
If k : even & $k > 2$, or k : odd & $k > \min\{k \mid 1_k \in D(k)\} =: 2n_D - 1$, we have



Consider the case k is odd, $k \neq 2n_D - 1$. We have the following inequalities among partitions.



By block-stable property, $\mathbf{1}_{k-2} \otimes \in D$.

Classification

D: blockwise category of interval partitions
$L_D := \{k \in \mathbb{N} \mid \mathbf{1}_k \in D(k)\}$

$$l_D := \sup\{l \in \mathbb{N} \mid 2l \in L_D\}.$$

$$m_D := \begin{cases} \sup\{m \in \mathbb{N} \mid 2m-1 \in L_D\}, & \text{if } L_D \text{ contains some odd numbers,} \\ \infty, & \text{otherwise.} \end{cases}$$

$$n_D := \begin{cases} \min\{m \in \mathbb{N} \mid 2m-1 \in L_D\}, & \text{if } L_D \text{ contains some odd numbers,} \\ \infty, & \text{otherwise.} \end{cases}$$

By lemma, we have

1. $m_D - n_D \leq l_D$ if $n_D \neq \infty$.
2. $l_D \leq m_D + n_D - 1$.

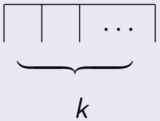
And D is determined by $l_D, m_D$and n_D.

A Boolean analogue of free quantum groups

Definition

D : a blockwise category of interval partitions.
$C(G_n^D) :=$ $*$-algebra generated by p, $u_{ij} (1 \le i, j \le n)$ with

$$p = p^* = p^2, u_{ij}^* = u_{ij}$$

and the following relations:
for any k with $\mathbf{1}_k :=$  $\in D(k)$,

$$\sum_{i=1}^{n} u_{ij_1} \cdots u_{ij_k} p = \begin{cases} p, & j_1 = \cdots = j_k, \\ 0, & \text{otherwise}, \end{cases}$$

$$\sum_{j=1}^{n} u_{i_1 j} \cdots u_{i_k j} p = \begin{cases} p, & i_1 = \cdots = i_k, \\ 0, & \text{otherwise}. \end{cases}$$

Notations on $C(G_n^D)$

Set a *-hom $\Delta\colon C(G_n^D) \to C(G_n^D) \otimes C(G_n^D)$ by

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj},$$
$$\Delta(p) = p \otimes p.$$

Δ is a coproduct: $(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta$.
Set $\mathscr{P}_\infty^o$:= the *-algebra of all nonunital polynomials in noncommutative countably infinite many variables $(X_j)_{j\in\mathbb{N}}$.
We can define a linear map $\Psi_n\colon \mathscr{P}_\infty^o \to \mathscr{P}_\infty^o \otimes C(G_n^D)$ as the extension of

$$\Psi_n(X_{j_1}\cdots X_{j_k}) := \sum_{\mathbf{i}\in[n]^k} X_{i_1}\cdots X_{i_k} \otimes p u_{i_1 j_1}\cdots u_{i_k j_k} p,\ \mathbf{j} \in [n]^k$$

Ψ_n is a coaction, that is,

$$(\Psi_n \otimes \mathrm{id}) \circ \Psi_n = (\mathrm{id} \otimes \Delta) \circ \Psi_n.$$

Fixed point algebra

Denote by $\mathscr{P}^{\Psi_n}$ the fixed point algebra:

$$\mathscr{P}^{\Psi_n} := \{f \in \mathscr{P}^o_\infty \mid f = f \otimes p\}.$$

We have

$$\mathscr{P}^{\Psi_n} = \mathrm{Span}\{X_\pi \in \mathscr{P}^o_\infty \mid \pi \in D(k), k \in \mathbb{N}\},$$

where $X_\pi := \sum_{\substack{\mathbf{j}\in[n]^k \\ \pi \le \ker \mathbf{j}}} X_{j_1} \cdots X_{j_k}$. By this representation of $\mathscr{P}^{\Psi_n}$, there is a functional h on the subspace S_n^D satisfying

$$(\mathrm{id} \otimes h)\Delta = (h \otimes \mathrm{id})\Delta = h.$$

Define a linear map $E_n \colon \mathscr{P}^o_\infty \to \mathscr{P}^{\Psi_n}$ by $E_n := (\mathrm{id} \otimes h) \circ \Psi_n$.

Invariance

(M, φ) : a v.N.algebra & a nondegenerate normal state.

Definition

$(x_j \in M_{s.a.})_{j \in \mathbb{N}}$ is said to have G^D-invariant joint distribution if

$$(\varphi \circ ev_x \otimes id) \circ \Psi_n = \varphi \circ ev_x \otimes p.$$

Main Theorem

Theorem

(M, φ) : a v.N.algebra & a nondegenerate normal state.
$x_j \in M_{s.a.}$, $j \in \mathbb{N}$ with $M = W^*(ev_x(\mathscr{P}^o_\infty))$
For any blockwise category of interval partitions D, TFAE.

1. *The joint distribution of $(x_j)_{j\in\mathbb{N}}$ is G^D-invariant.*
2. *$(x_j)_{j\in\mathbb{N}}$ is Boolean i.i.d. over tail,*
   & for any k with $1_k \in D(k)$, $K_k^{E_{\mathrm{tail}}}[x_1 b_1, x_1 b_2, \dots, x_1] = 0$
   , $b_1, \cdots, b_k \in M_{\mathrm{tail}} \cup \{1\}$.

In particular,

Symmetries	Categories of partitions	Distributions
G_n^I	I	Boolean i.i.d. over tail (*)
$G_n^{I_2}$	I_2	(*) & centered Bernoulli
$G_n^{I_b}$	I_b	(*) & Bernoulli
$G_n^{I_h}$	I_h	(*) & even

Strategy

Assume the joint distribution of $(x_j)_{j\in\mathbb{N}}$ is G^D-invariant.
Since G^D-invariance implies $\mathcal{B}_s$-invariance, there exist a normal c.e.
$E_{tail} : M \to M_{tail}$ given by $E_{tail} = e_{tail}(\cdot)e_{tail}$.
ISTS for any $b_0, \ldots, b_k \in M_{\rm tail} \cup \{1\}$, $\mathbf{j} \in [n]^k$, and $k \in \mathbb{N}$,

$$E_{\rm tail}[x_{j_1} b_1 x_{j_2} b_2 \cdots b_{k-1} x_{j_k}] = \sum_{\substack{\sigma\in D(k)\\ \sigma\le\ker\mathbf{j}}} K_\sigma^{E_{\rm tail}}[x_1 b_1, x_1 b_2, \ldots, x_1].$$

Main strategy of the proof:

1. Examine E_{tail} can be approximated by $E_n := (id \otimes h)\Psi_n$
2. Use Weingarten estimate
   If D is blockwise then $\mu_{D(k)} = \mu_{I(k)}$. By using this,

$$h(pu_{i_1 j_1} \cdots u_{i_k j_k} p) = \sum_{\substack{\pi,\sigma\in D(k)\\ \pi\le\ker\mathbf{i}\\ \sigma\le\ker\mathbf{j}}} \frac{1}{n^{|\pi|}}[\mu_{I(k)}(\pi,\sigma) + O(\frac{1}{n})] \text{ (as } n \to \infty)$$

3. Apply moments-cumulants formula.

Difficulty 1 : Coaction is non-multiplicative

Since the coaction Ψ_n is non-multiplicative :

$$\Psi_n(f(X)g(X)) \neq \Psi_n(f(X))\Lambda_n(g(X)),$$

there exist $b_1, \ldots, b_{k-1} \in \mathscr{P}^{\Psi_n}$ with

$$\Psi_n[X_{j_1} b_1 X_{j_2} b_2 \cdots b_{k-1} X_{j_k}] \neq \sum_{\mathbf{i} \in [n]^k} X_{i_1} b_1 X_{i_2} b_2 \cdots b_{k-1} X_{i_k} \otimes p u_{i_1 j_1} \ldots u_{i_k j_k} p,$$

So it is difficult to approximate
$E_{\mathrm{tail}}[x_{j_1} b_1 \cdots b_{k-1} x_{j_k}]$ by $E_n[X_{j_1} b_1 X_{j_2} b_2 \cdots b_{k-1} X_{j_k}]$.
idea: By block-stable condition, and since E_{tail} satisfies $E_{tail} = e_{tail}(\cdot)e_{tail}$, the following holds; Assume for any $\mathbf{j} \in [n]^k$ and $k \in \mathbb{N}$,

$$E_{\mathrm{tail}}[x_{j_1} \cdots x_{j_k}] = \sum_{\substack{\sigma \in D(k) \\ \sigma \leq \ker \mathbf{j}}} K_\sigma^{E_{\mathrm{tail}}}[x_1, \ldots, x_1].$$

Then for any $b_0, \ldots, b_k \in M_{\mathrm{tail}} \cup \{1\}$, $\mathbf{j} \in [n]^k$, and $k \in \mathbb{N}$,

$$E_{\mathrm{tail}}[x_{j_1} b_1 x_{j_2} b_2 \cdots b_{k-1} x_{j_k}] = \sum_{\substack{\sigma \in D(k) \\ \sigma \leq \ker \mathbf{j}}} K_\sigma^{E_{\mathrm{tail}}}[x_1 b_1, x_1 b_2, \ldots, x_1].$$

Difficulty 2

Difficulty 2 : As the state φ is non-faithful, we cannot define E_n on M and cannot approximate E_{tail} by E_n.
idea:
$e_n :=$ the orthogonal projection onto $\overline{\mathrm{ev}_x(\mathscr{P}^{\Psi_n})\Omega_\varphi}$. If we prove
$L^2\text{-}\lim_n ev_x(E_n[X_{j_1}X_{j_2}\cdots X_{j_k}])\Omega_\varphi = E_{tail}[x_{j_1}x_{j_2}\cdots x_{j_k}]\Omega_\varphi$ ($\mathbf{j} \in [n]^k, k \in \mathbb{N}$)
Then $s\text{-}\lim e_n = e_{tail}$ and hence

$$s - \lim_n ev_x(E_n[X_{j_1}X_{j_2}\cdots X_{j_k}])e_n = E_{tail}[x_{j_1}x_{j_2}\cdots x_{j_k}](\mathbf{j} \in [n]^k, k \in \mathbb{N})$$

Difficulties and key ideas

Difficulty1 : As the state is non-faithful, we cannot define E_n on M and cannot approximate E_{tail} by E_n.
Idea : ISTS

$$\lim_{n\to\infty} E_{tail}[x_{j_1} x_{j_2} \cdots x_{j_k}] = L^2 - \lim_{n\to\infty} \mathrm{ev}_x \circ E_n[X_{j_1} X_{j_2} \cdots X_{j_k}].$$

Difficulty2 : Coactions are non-multiplicative. Hence it is difficult to estimate $E_{\mathrm{tail}}[x_{j_1} b_1 x_{j_2} b_2 \cdots b_{k-1} x_{j_k}]$ $(b_0, \ldots, b_k \in M_{\mathrm{tail}} \cup \{1\})$.
Idea : ISTS

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C*-closure

Free case: By Tannaka-Klein duality for compact quantum groups,
Free quantum groups $A_x \Longleftrightarrow NC_x$.

$$A_x(n) = C^*_{univ}(u = (u_{ij}) \mid u^t u = {}^t u u = 1)/\text{relations implied by } NC_x$$
$$\mathbb{C}\langle X_j \mid j \in \mathbb{N}\rangle^{\Psi_n^{A_x}} = \text{Span}\{X_\pi \in \mathscr{P}^o_\infty \mid \pi \in NC_x\}.$$

Boolean case: $C^*_{univ}(p, u = (u_{ij}) \mid \text{relations implied by } D)$ can be **ill-defined.**
Liu: $B_o(n) := C^*_{univ}(p, u = (u_{ij}) \mid p = p^* = p^2, u^t u p = {}^t u u = p, \|u\| \leq 1)$
It is not clear

$$\mathscr{P}^{\Psi_n^{B_o}} \stackrel{?}{=} \text{Span}\{X_\pi \in \mathscr{P}^o_\infty \mid \pi \in I_2\}.$$

Hence h and E_n can be changed, it is not obvious that our strategy works well for $B_o(n)$.

Summary

Aim : Prove general Boolean de Finetti theorem.
Our strategy : Find a nice class of interval partitions and use BCS's framework.
Key condition: D is **blockwise** i.e. block-stable and closed under taking an interval in I. Second condition implies

$$\mu_{D(k)}(\pi,\sigma) = \mu_{I(k)}(\pi,\sigma), \pi,\sigma \in D(k).$$

Difficulty1 : As the state φ is **non-faithful**, it is difficult to define E_n on M and approximate E_{tail} by E_n.
Idea : ISTS

$$E_{tail}[x_{j_1}x_{j_2}\cdots x_{j_k}] = L^2 - \lim_{n\to\infty} \mathrm{ev}_x \circ E_n[X_{j_1}X_{j_2}\cdots X_{j_k}].$$

Difficulty2 : Coactions are non-multiplicative. Hence it is difficult to estimate $E_{\mathrm{tail}}[x_{j_1}b_1x_{j_2}b_2\cdots b_{k-1}x_{j_k}]$ $(b_0,\ldots,b_k \in M_{\mathrm{tail}} \cup \{1\})$.
Idea : By block-stable condition, ISTS

$$E_{\mathrm{tail}}[x_{j_1}\cdots x_{j_k}] = \sum_{\substack{\sigma\in D(k)\\ \sigma\le\ker\mathbf{j}}} K_\sigma^{E_{\mathrm{tail}}}[x_1,\ldots,x_1].$$