The fundamental theorem of arithmetic for metric measure spaces

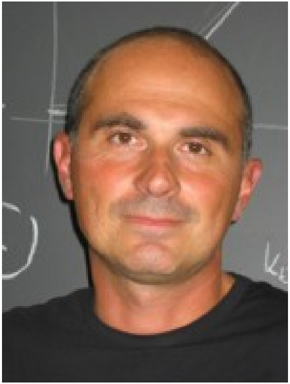
Steven N. Evans

U.C. Berkeley

March, 2016

Collaborator

Ilya Molchanov
Bern
To appear in *Transactions of the American Mathematical Society*
Available at http://arxiv.org/abs/1401.7052



Cartesian product

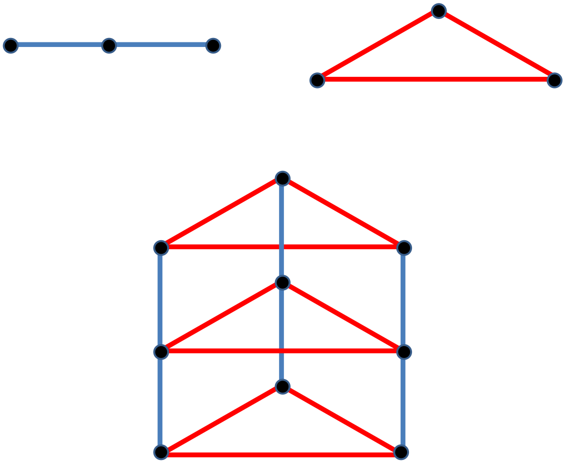


Figure: The Cartesian product of two graphs.

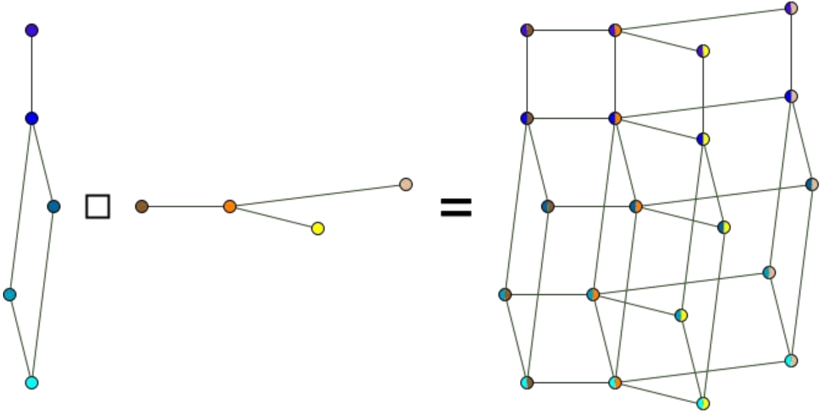


Figure: The Cartesian product of two more interesting graphs (courtesy of *Wikipedia*).

- Formally, the Cartesian product $G\Box H$ of two graphs G and H with vertex sets $V(G)$ and $V(H)$ and edge sets $E(G)$ and $E(H)$ is the graph with vertex set $V(G\Box H) := V(G) \times V(H)$ and edge set

$$\begin{aligned} E(G\Box H) := \{&((g',h),(g'',h)) : (g',g'') \in E(G),\, h \in V(H)\} \\ &\cup \{((g,h'),(g,h'')) : g \in V(G),\, (h',h'') \in E(H)\}. \end{aligned}$$

- This operation is commutative and associative with the trivial graph as identity element if we treat isomorphic graphs as being equal.

Cartesian product

- Formally, the Cartesian product $G \square H$ of two graphs G and H with vertex sets $V(G)$ and $V(H)$ and edge sets $E(G)$ and $E(H)$ is the graph with vertex set $V(G \square H) := V(G) \times V(H)$ and edge set

$$\begin{aligned} E(G \square H) := & \{((g', h), (g'', h)) : (g', g'') \in E(G),\ h \in V(H)\} \\ & \cup \{((g, h'), (g, h'')) : g \in V(G),\ (h', h'') \in E(H)\}. \end{aligned}$$

- This operation is commutative and associative with the trivial graph as identity element if we treat isomorphic graphs as being equal.

Cartesian product

- A nontrivial graph is irreducible if it is not the Cartesian product of two nontrivial graphs.
- Sabidussi (1960) showed that any finite graph is a Cartesian product of irreducible graphs and the factorization is unique up to order.
- Factoring graphs and, more generally, embedding them in Cartesian products is widely studied in computer science following Graham and Winkler (1984, 1985).

Cartesian product

- A nontrivial graph is irreducible if it is not the Cartesian product of two nontrivial graphs.
- Sabidussi (1960) showed that any finite graph is a Cartesian product of irreducible graphs and the factorization is unique up to order.
- Factoring graphs and, more generally, embedding them in Cartesian products is widely studied in computer science following Graham and Winkler (1984, 1985).

Cartesian product

- A nontrivial graph is irreducible if it is not the Cartesian product of two nontrivial graphs.
- Sabidussi (1960) showed that any finite graph is a Cartesian product of irreducible graphs and the factorization is unique up to order.
- Factoring graphs and, more generally, embedding them in Cartesian products is widely studied in computer science following Graham and Winkler (1984, 1985).

Shortest path metric

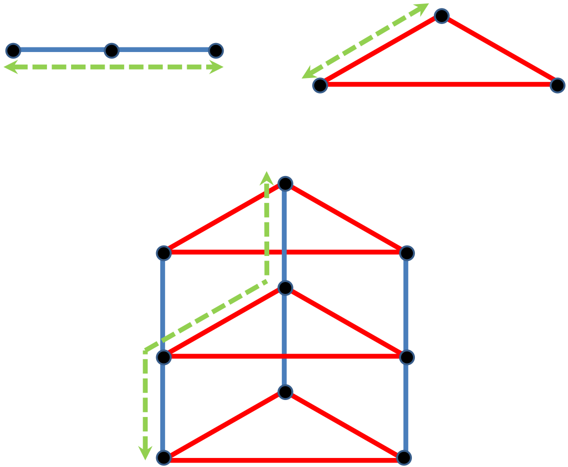


Figure: A shortest path between two points in the Cartesian product of two graphs.

Shortest path metric

- If two connected finite graphs G and H are equipped with the shortest path metrics r_G and r_H, then the shortest path metric on the Cartesian product is given by

$$r_{G \times H} = r_G \bigoplus r_H,$$

  where

$$(r_G \bigoplus r_H)((g', h'), (g'', h'')) := r_G(g', g'') + r_H(h', h''), \quad (g', h'), (g'', h'') \in G \times H.$$

- What happens if we extend this binary operation to more general metric spaces?

Shortest path metric

- If two connected finite graphs G and H are equipped with the shortest path metrics r_G and r_H, then the shortest path metric on the Cartesian product is given by
$$r_{G \times H} = r_G \bigoplus r_H,$$
where
$$(r_G \bigoplus r_H)((g', h'), (g'', h'')) := r_G(g', g'') + r_H(h', h''), \quad (g', h'), \ (g'', h'') \in G \times H.$$
- What happens if we extend this binary operation to more general metric spaces?

Cartesian product of two intervals and the Manhattan/taxi-cab/ℓ^1 metric

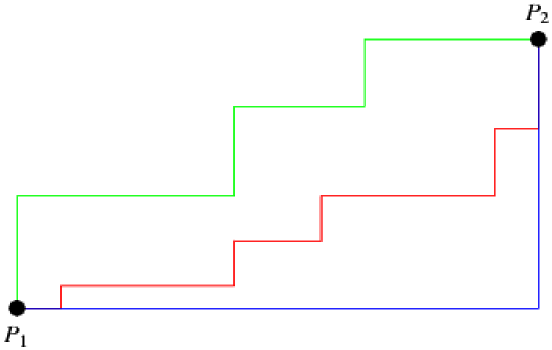


Figure: Equipping the Cartesian product of two intervals with the sum of the usual metrics gives a rectangle equipped with the Manhattan or taxi-cab metric (courtesy of *Wolfram*).

Irreducibles

The metric space (X, r_X) is irreducible if there is no nontrivial factorization

$$(X, r_X) = (Y \times Z, r_Y \oplus r_Z).$$

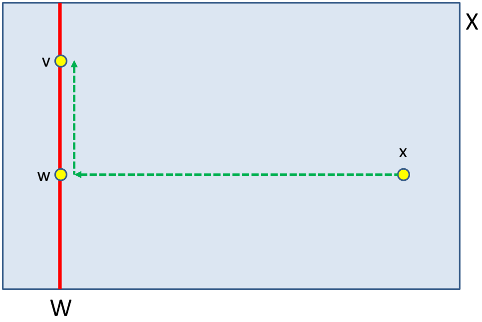
Uniqueness of factorization into irreducibles

If a metric space is isometric to a product of finitely many irreducible metric spaces, then this factorization is unique up to the order of the factors – Tardif (1992).

A little about Tardif's proof

Tardif uses ideas/results from the world of median algebras, Chebyshev sets, gated spaces from Isbell (1980), Helíková (1983), Dress & Scharlau (1987).

A subset W of a metric space (X, r_X) is gated if for each $x \in X$ there is a (necessarily unique) $w \in W$ such that $r_X(x, v) = r_X(x, w) + r_X(w, v)$ for all $v \in W$ (for any $v \in W$, we can always choose a shortest path from x to v that passes through the gate w).



Limitations of factorization

- There are certainly compact metric spaces that are not isometric to a finite product of finitely many irreducible compact metric spaces (e.g. $X := \prod_{k\in\mathbb{N}}[0, a_k]$, where $\sum_{k\in\mathbb{N}} a_k < \infty$, with $r_X(x', x'') := \sum_{k\in\mathbb{N}} |x'_k - x''_k|$).
- Are there (unique) factorizations using some sort of infinite product? What does this mean?
- The study of this binary operation seems to be generally rather difficult.

Limitations of factorization

- There are certainly compact metric spaces that are not isometric to a finite product of finitely many irreducible compact metric spaces (e.g. $X := \prod_{k \in \mathbb{N}} [0, a_k]$, where $\sum_{k \in \mathbb{N}} a_k < \infty$, with $r_X(x', x'') := \sum_{k \in \mathbb{N}} |x'_k - x''_k|$).
- Are there (unique) factorizations using some sort of infinite product? What does this mean?
- The study of this binary operation seems to be generally rather difficult.

Limitations of factorization

- There are certainly compact metric spaces that are not isometric to a finite product of finitely many irreducible compact metric spaces (e.g. $X := \prod_{k\in\mathbb{N}}[0, a_k]$, where $\sum_{k\in\mathbb{N}} a_k < \infty$, with $r_X(x', x'') := \sum_{k\in\mathbb{N}} |x'_k - x''_k|$).
- Are there (unique) factorizations using some sort of infinite product? What does this mean?
- The study of this binary operation seems to be generally rather difficult.

- A metric measure space is just a complete separable metric space (X, r_X) equipped with a probability measure μ_X that has full support.
- Two such spaces are equivalent if they are isometric as metric spaces via an isometry that maps the probability measure on the first space to the probability measure on the second.
- Denote by $\mathbb{M}$ the set of such equivalence classes.
- We do not distinguish between an equivalence class $\mathcal{X} \in \mathbb{M}$ and a representative triple (X, r_X, μ_X).

Metric measure spaces

- A metric measure space is just a complete separable metric space (X, r_X) equipped with a probability measure μ_X that has full support.
- Two such spaces are equivalent if they are isometric as metric spaces via an isometry that maps the probability measure on the first space to the probability measure on the second.
- Denote by $\mathbb{M}$ the set of such equivalence classes.
- We do not distinguish between an equivalence class $\mathcal{X} \in \mathbb{M}$ and a representative triple (X, r_X, μ_X).

- A metric measure space is just a complete separable metric space (X, r_X) equipped with a probability measure μ_X that has full support.
- Two such spaces are equivalent if they are isometric as metric spaces via an isometry that maps the probability measure on the first space to the probability measure on the second.
- Denote by $\mathbb{M}$ the set of such equivalence classes.
- We do not distinguish between an equivalence class $\mathcal{X} \in \mathbb{M}$ and a representative triple (X, r_X, μ_X).

Metric measure spaces

- A metric measure space is just a complete separable metric space (X, r_X) equipped with a probability measure μ_X that has full support.
- Two such spaces are equivalent if they are isometric as metric spaces via an isometry that maps the probability measure on the first space to the probability measure on the second.
- Denote by $\mathbb{M}$ the set of such equivalence classes.
- We do not distinguish between an equivalence class $\mathcal{X} \in \mathbb{M}$ and a representative triple (X, r_X, μ_X).

When are two metric measures spaces equivalent?

- Gromov and Vershik showed that, in probabilist-speak, a metric measure space (X, r_X, μ_X) is uniquely determined by the probability distribution of the infinite random matrix of distances

$$(r_X(\xi_i, \xi_j))_{(i,j)\in\mathbb{N}\times\mathbb{N}},$$

  where $(\xi_k)_{k\in\mathbb{N}}$ is an i.i.d. sample of points in X with common probability distribution μ_X.

- In non-probabilist-speak, (X, r_X, μ_X) is determined by the push-forward of the probability measure $\mu_X^{\otimes\mathbb{N}}$ by the function

$$X^{\mathbb{N}} \ni (x_k)_{k\in\mathbb{N}} \mapsto (r_X(x_i, x_j))_{i,j\in\mathbb{N}\times\mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}\times\mathbb{N}}.$$

- This concise condition for equivalence makes metric measure spaces considerably easier to study than complete separable metric spaces *per se*.

When are two metric measures spaces equivalent?

- Gromov and Vershik showed that, in probabilist-speak, a metric measure space (X, r_X, μ_X) is uniquely determined by the probability distribution of the infinite random matrix of distances

$$(r_X(\xi_i, \xi_j))_{(i,j)\in\mathbb{N}\times\mathbb{N}},$$

  where $(\xi_k)_{k\in\mathbb{N}}$ is an i.i.d. sample of points in X with common probability distribution μ_X.

- In non-probabilist-speak, (X, r_X, μ_X) is determined by the push-forward of the probability measure $\mu_X^{\otimes\mathbb{N}}$ by the function

$$X^{\mathbb{N}} \ni (x_k)_{k\in\mathbb{N}} \mapsto (r_X(x_i, x_j))_{i,j\in\mathbb{N}\times\mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}\times\mathbb{N}}.$$

- This concise condition for equivalence makes metric measure spaces considerably easier to study than complete separable metric spaces *per se*.

- Gromov and Vershik showed that, in probabilist-speak, a metric measure space (X, r_X, μ_X) is uniquely determined by the probability distribution of the infinite random matrix of distances

$$(r_X(\xi_i, \xi_j))_{(i,j)\in\mathbb{N}\times\mathbb{N}},$$

  where $(\xi_k)_{k\in\mathbb{N}}$ is an i.i.d. sample of points in X with common probability distribution μ_X.
- In non-probabilist-speak, (X, r_X, μ_X) is determined by the push-forward of the probability measure $\mu_X^{\otimes\mathbb{N}}$ by the function

$$X^{\mathbb{N}} \ni (x_k)_{k\in\mathbb{N}} \mapsto (r_X(x_i, x_j))_{i,j\in\mathbb{N}\times\mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}\times\mathbb{N}}.$$

- This concise condition for equivalence makes metric measure spaces considerably easier to study than complete separable metric spaces *per se*.

A binary operation

- Given two elements $\mathcal{Y} = (Y, r_Y, \mu_Y)$ and $\mathcal{Z} = (Z, r_Z, \mu_Z)$ of $\mathbb{M}$, let $\mathcal{Y} \boxplus \mathcal{Z}$ be $\mathcal{X} = (X, r_X, \mu_X) \in \mathbb{M}$, where
  - $X := Y \times Z$,
  - $r_X := r_Y \oplus r_Z$, where $(r_Y \oplus r_Z)((y', z'), (y'', z'')) = r_Y(y', y'') + r_Z(z', z'')$ for $(y', z'), (y'', z'') \in Y \times Z$),
  - $\mu_X := \mu_Y \otimes \mu_Z$.
- This binary operation is associative and commutative.
- The isometry class of metric measure spaces $\mathcal{E}$ that each consist of a single point with the only possible metric and probability measure on them is the identity element.
- Thus, $(\mathbb{M}, \boxplus)$ is a commutative semigroup with an identity (i.e. a monoid).

A binary operation

- Given two elements $\mathcal{Y} = (Y, r_Y, \mu_Y)$ and $\mathcal{Z} = (Z, r_Z, \mu_Z)$ of $\mathbb{M}$, let $\mathcal{Y} \boxplus \mathcal{Z}$ be $\mathcal{X} = (X, r_X, \mu_X) \in \mathbb{M}$, where
  - $X := Y \times Z$,
  - $r_X := r_Y \oplus r_Z$, where $(r_Y \oplus r_Z)((y', z'), (y'', z'')) = r_Y(y', y'') + r_Z(z', z'')$ for $(y', z'), (y'', z'') \in Y \times Z)$,
  - $\mu_X := \mu_Y \otimes \mu_Z$.
- This binary operation is associative and commutative.
- The isometry class of metric measure spaces $\mathcal{E}$ that each consist of a single point with the only possible metric and probability measure on them is the identity element.
- Thus, $(\mathbb{M}, \boxplus)$ is a commutative semigroup with an identity (i.e. a monoid).

A binary operation

- Given two elements $\mathcal{Y} = (Y, r_Y, \mu_Y)$ and $\mathcal{Z} = (Z, r_Z, \mu_Z)$ of $\mathbb{M}$, let $\mathcal{Y} \boxplus \mathcal{Z}$ be $\mathcal{X} = (X, r_X, \mu_X) \in \mathbb{M}$, where
  - $X := Y \times Z$,
  - $r_X := r_Y \oplus r_Z$, where $(r_Y \oplus r_Z)((y', z'), (y'', z'')) = r_Y(y', y'') + r_Z(z', z'')$ for $(y', z'), (y'', z'') \in Y \times Z$),
  - $\mu_X := \mu_Y \otimes \mu_Z$.
- This binary operation is associative and commutative.
- The isometry class of metric measure spaces $\mathcal{E}$ that each consist of a single point with the only possible metric and probability measure on them is the identity element.
- Thus, $(\mathbb{M}, \boxplus)$ is a commutative semigroup with an identity (i.e. a monoid).

A binary operation

- Given two elements $\mathcal{Y} = (Y, r_Y, \mu_Y)$ and $\mathcal{Z} = (Z, r_Z, \mu_Z)$ of $\mathbb{M}$, let $\mathcal{Y} \boxplus \mathcal{Z}$ be $\mathcal{X} = (X, r_X, \mu_X) \in \mathbb{M}$, where
  - $X := Y \times Z$,
  - $r_X := r_Y \oplus r_Z$, where $(r_Y \oplus r_Z)((y', z'), (y'', z'')) = r_Y(y', y'') + r_Z(z', z'')$ for $(y', z'), (y'', z'') \in Y \times Z$),
  - $\mu_X := \mu_Y \otimes \mu_Z$.
- This binary operation is associative and commutative.
- The isometry class of metric measure spaces $\mathcal{E}$ that each consist of a single point with the only possible metric and probability measure on them is the identity element.
- Thus, $(\mathbb{M}, \boxplus)$ is a commutative semigroup with an identity (i.e. a monoid).

To paraphrase T.W. Körner

"The ease with which we proved [the central limit theorem] explains why Fourier analysis plays a rôle in probability theory that in other branches of mathematics is played by thought."

Semicharacters

- A semicharacter is a map $\chi : \mathbb{M} \to [0,1]$ such that $\chi(\mathcal{Y} \boxplus \mathcal{Z}) = \chi(\mathcal{Y})\chi(\mathcal{Z})$ for all $\mathcal{Y}, \mathcal{Z} \in \mathbb{M}$.
- Denote by $\mathbb{A}$ the family of arrays of the form $A = (a_{ij})_{1 \leqslant i < j \leqslant n} \in \mathbb{R}_+^{\binom{n}{2}}$ for $n \in \mathbb{N}$.
- For each $A \in \mathbb{A}$ define a semicharacter χ_A by

$$\chi_A((X, r_X, \mu_X)) := \int_{X^n} \exp\left(-\sum_{1 \leqslant i < j \leqslant n} a_{ij} r_X(x_i, x_j)\right) \mu_X^{\otimes n}(dx).$$

- Two elements $\mathcal{X}, \mathcal{Y} \in \mathbb{M}$ are equal if and only if $\chi_A(\mathcal{X}) = \chi_A(\mathcal{Y})$ for all $A \in \mathbb{A}$.

Semicharacters

- A semicharacter is a map $\chi : \mathbb{M} \to [0,1]$ such that $\chi(\mathcal{Y} \boxplus \mathcal{Z}) = \chi(\mathcal{Y})\chi(\mathcal{Z})$ for all $\mathcal{Y}, \mathcal{Z} \in \mathbb{M}$.
- Denote by $\mathbb{A}$ the family of arrays of the form $A = (a_{ij})_{1 \leqslant i < j \leqslant n} \in \mathbb{R}_+^{\binom{n}{2}}$ for $n \in \mathbb{N}$.
- For each $A \in \mathbb{A}$ define a semicharacter χ_A by

$$\chi_A((X, r_X, \mu_X)) := \int_{X^n} \exp\left(-\sum_{1 \leqslant i < j \leqslant n} a_{ij} r_X(x_i, x_j)\right) \mu_X^{\otimes n}(dx).$$

- Two elements $\mathcal{X}, \mathcal{Y} \in \mathbb{M}$ are equal if and only if $\chi_A(\mathcal{X}) = \chi_A(\mathcal{Y})$ for all $A \in \mathbb{A}$.

Semicharacters

- A semicharacter is a map $\chi : \mathbb{M} \to [0,1]$ such that $\chi(\mathcal{Y} \boxplus \mathcal{Z}) = \chi(\mathcal{Y})\chi(\mathcal{Z})$ for all $\mathcal{Y}, \mathcal{Z} \in \mathbb{M}$.
- Denote by $\mathbb{A}$ the family of arrays of the form $A = (a_{ij})_{1 \leqslant i < j \leqslant n} \in \mathbb{R}_+^{\binom{n}{2}}$ for $n \in \mathbb{N}$.
- For each $A \in \mathbb{A}$ define a semicharacter χ_A by

$$\chi_A((X, r_X, \mu_X)) := \int_{X^n} \exp\left(-\sum_{1 \leqslant i < j \leqslant n} a_{ij} r_X(x_i, x_j)\right) \mu_X^{\otimes n}(dx).$$

- Two elements $\mathcal{X}, \mathcal{Y} \in \mathbb{M}$ are equal if and only if $\chi_A(\mathcal{X}) = \chi_A(\mathcal{Y})$ for all $A \in \mathbb{A}$.

Semicharacters

- A semicharacter is a map $\chi : \mathbb{M} \to [0,1]$ such that $\chi(\mathcal{Y} \boxplus \mathcal{Z}) = \chi(\mathcal{Y})\chi(\mathcal{Z})$ for all $\mathcal{Y}, \mathcal{Z} \in \mathbb{M}$.
- Denote by $\mathbb{A}$ the family of arrays of the form $A = (a_{ij})_{1 \leqslant i < j \leqslant n} \in \mathbb{R}_+^{\binom{n}{2}}$ for $n \in \mathbb{N}$.
- For each $A \in \mathbb{A}$ define a semicharacter χ_A by

$$\chi_A((X, r_X, \mu_X)) := \int_{X^n} \exp\left(- \sum_{1 \leqslant i < j \leqslant n} a_{ij} r_X(x_i, x_j) \right) \mu_X^{\otimes n}(dx).$$

- Two elements $\mathcal{X}, \mathcal{Y} \in \mathbb{M}$ are equal if and only if $\chi_A(\mathcal{X}) = \chi_A(\mathcal{Y})$ for all $A \in \mathbb{A}$.

- Equip $\mathbb{M}$ with the Gromov-Prohorov metric of Greven, Pfaffelhuber & Winter (2009). Two elements of $\mathbb{M}$ are close if their random distance matrices are close in distribution.
- The space $(\mathbb{M}, d_{\mathrm{GPr}})$ is complete and separable (e.g. finite metric spaces with rational distances are dense).
- $d_{\mathrm{GPr}}(\mathcal{X}_1 \boxplus \mathcal{X}_2, \mathcal{Y}_1 \boxplus \mathcal{Y}_2) \leqslant d_{\mathrm{GPr}}(\mathcal{X}_1, \mathcal{Y}_1) + d_{\mathrm{GPr}}(\mathcal{X}_2, \mathcal{Y}_2)$ and so $(\mathcal{X}, \mathcal{Y}) \mapsto \mathcal{X} \boxplus \mathcal{Y}$ is continuous.
- $\lim_{n\to\infty} \mathcal{X}_n = \mathcal{X}$ if and only if $\lim_{n\to\infty} \chi_A(\mathcal{X}_n) = \chi_A(\mathcal{X})$ for all $A \in \mathbb{A}$.
- Note: Knowing that $\lim_{n\to\infty} \chi_A(\mathcal{X}_n)$ exists for all $A \in \mathbb{A}$ does not imply that $\lim_{n\to\infty} \mathcal{X}_n$ exists.

Putting a metric on $\mathbb{M}$

- Equip $\mathbb{M}$ with the Gromov-Prohorov metric of Greven, Pfaffelhuber & Winter (2009). Two elements of $\mathbb{M}$ are close if their random distance matrices are close in distribution.
- The space $(\mathbb{M}, d_{\mathrm{GPr}})$ is complete and separable (e.g. finite metric spaces with rational distances are dense).
- $d_{\mathrm{GPr}}(\mathcal{X}_1 \boxplus \mathcal{X}_2, \mathcal{Y}_1 \boxplus \mathcal{Y}_2) \leqslant d_{\mathrm{GPr}}(\mathcal{X}_1, \mathcal{Y}_1) + d_{\mathrm{GPr}}(\mathcal{X}_2, \mathcal{Y}_2)$ and so $(\mathcal{X}, \mathcal{Y}) \mapsto \mathcal{X} \boxplus \mathcal{Y}$ is continuous.
- $\lim_{n\to\infty} \mathcal{X}_n = \mathcal{X}$ if and only if $\lim_{n\to\infty} \chi_A(\mathcal{X}_n) = \chi_A(\mathcal{X})$ for all $A \in \mathbb{A}$.
- Note: Knowing that $\lim_{n\to\infty} \chi_A(\mathcal{X}_n)$ exists for all $A \in \mathbb{A}$ does not imply that $\lim_{n\to\infty} \mathcal{X}_n$ exists.

Putting a metric on $\mathbb{M}$

- Equip $\mathbb{M}$ with the Gromov-Prohorov metric of Greven, Pfaffelhuber & Winter (2009). Two elements of $\mathbb{M}$ are close if their random distance matrices are close in distribution.
- The space $(\mathbb{M}, d_{\mathrm{GPr}})$ is complete and separable (e.g. finite metric spaces with rational distances are dense).
- $d_{\mathrm{GPr}}(\mathcal{X}_1 \boxplus \mathcal{X}_2, \mathcal{Y}_1 \boxplus \mathcal{Y}_2) \leqslant d_{\mathrm{GPr}}(\mathcal{X}_1, \mathcal{Y}_1) + d_{\mathrm{GPr}}(\mathcal{X}_2, \mathcal{Y}_2)$ and so $(\mathcal{X}, \mathcal{Y}) \mapsto \mathcal{X} \boxplus \mathcal{Y}$ is continuous.
- $\lim_{n\to\infty} \mathcal{X}_n = \mathcal{X}$ if and only if $\lim_{n\to\infty} \chi_A(\mathcal{X}_n) = \chi_A(\mathcal{X})$ for all $A \in \mathbb{A}$.
- Note: Knowing that $\lim_{n\to\infty} \chi_A(\mathcal{X}_n)$ exists for all $A \in \mathbb{A}$ does not imply that $\lim_{n\to\infty} \mathcal{X}_n$ exists.

Putting a metric on $\mathbb{M}$

- Equip $\mathbb{M}$ with the Gromov-Prohorov metric of Greven, Pfaffelhuber & Winter (2009). Two elements of $\mathbb{M}$ are close if their random distance matrices are close in distribution.
- The space $(\mathbb{M}, d_{\mathrm{GPr}})$ is complete and separable (e.g. finite metric spaces with rational distances are dense).
- $d_{\mathrm{GPr}}(\mathcal{X}_1 \boxplus \mathcal{X}_2, \mathcal{Y}_1 \boxplus \mathcal{Y}_2) \leqslant d_{\mathrm{GPr}}(\mathcal{X}_1, \mathcal{Y}_1) + d_{\mathrm{GPr}}(\mathcal{X}_2, \mathcal{Y}_2)$ and so $(\mathcal{X}, \mathcal{Y}) \mapsto \mathcal{X} \boxplus \mathcal{Y}$ is continuous.
- $\lim_{n\to\infty} \mathcal{X}_n = \mathcal{X}$ if and only if $\lim_{n\to\infty} \chi_A(\mathcal{X}_n) = \chi_A(\mathcal{X})$ for all $A \in \mathbb{A}$.
- Note: Knowing that $\lim_{n\to\infty} \chi_A(\mathcal{X}_n)$ exists for all $A \in \mathbb{A}$ does not imply that $\lim_{n\to\infty} \mathcal{X}_n$ exists.

Putting a metric on $\mathbb{M}$

- Equip $\mathbb{M}$ with the Gromov-Prohorov metric of Greven, Pfaffelhuber & Winter (2009). Two elements of $\mathbb{M}$ are close if their random distance matrices are close in distribution.
- The space $(\mathbb{M}, d_{\mathrm{GPr}})$ is complete and separable (e.g. finite metric spaces with rational distances are dense).
- $d_{\mathrm{GPr}}(\mathcal{X}_1 \boxplus \mathcal{X}_2, \mathcal{Y}_1 \boxplus \mathcal{Y}_2) \leqslant d_{\mathrm{GPr}}(\mathcal{X}_1, \mathcal{Y}_1) + d_{\mathrm{GPr}}(\mathcal{X}_2, \mathcal{Y}_2)$ and so $(\mathcal{X}, \mathcal{Y}) \mapsto \mathcal{X} \boxplus \mathcal{Y}$ is continuous.
- $\lim_{n\to\infty} \mathcal{X}_n = \mathcal{X}$ if and only if $\lim_{n\to\infty} \chi_A(\mathcal{X}_n) = \chi_A(\mathcal{X})$ for all $A \in \mathbb{A}$.
- Note: Knowing that $\lim_{n\to\infty} \chi_A(\mathcal{X}_n)$ exists for all $A \in \mathbb{A}$ does not imply that $\lim_{n\to\infty} \mathcal{X}_n$ exists.

Putting a partial order on $\mathbb{M}$

- Define a partial order $\leq$ on $\mathbb{M}$ by declaring that $\mathcal{Y} \leq \mathcal{Z}$ if $\mathcal{Z} = \mathcal{Y} \boxplus \mathcal{X}$ for some $\mathcal{X} \in \mathbb{M}$. That is, $\mathcal{Y} \leq \mathcal{Z}$ if $\mathcal{Y}$ is a "divisor" of $\mathcal{Z}$.
- For any $\mathcal{Z} \in \mathbb{M}$, the set $\{\mathcal{Y} \in \mathbb{M} : \mathcal{Y} \leq \mathcal{Z}\}$ is compact.

Putting a partial order on $\mathbb{M}$

- Define a partial order $\leqslant$ on $\mathbb{M}$ by declaring that $\mathcal{Y} \leqslant \mathcal{Z}$ if $\mathcal{Z} = \mathcal{Y} \boxplus \mathcal{X}$ for some $\mathcal{X} \in \mathbb{M}$. That is, $\mathcal{Y} \leqslant \mathcal{Z}$ if $\mathcal{Y}$ is a "divisor" of $\mathcal{Z}$.
- For any $\mathcal{Z} \in \mathbb{M}$, the set $\{\mathcal{Y} \in \mathbb{M} : \mathcal{Y} \leqslant \mathcal{Z}\}$ is compact.

Cancellativity

- The commutative semigroup $(\mathbb{M}, \boxplus)$ is cancellative; that is, if $\mathcal{X}, \mathcal{Y}, \mathcal{Z}', \mathcal{Z}'' \in \mathbb{M}$ satisfy

  $$\mathcal{X} = \mathcal{Y} \boxplus \mathcal{Z}'$$

  and

  $$\mathcal{X} = \mathcal{Y} \boxplus \mathcal{Z}'',$$

  then

  $$\mathcal{Z}' = \mathcal{Z}''.$$

- This is because for all $A \in \mathbb{A}$

  $$\chi_A(\mathcal{Y})\chi_A(\mathcal{Z}') = \chi_A(X) = \chi_A(\mathcal{Y})\chi_A(\mathcal{Z}''),$$

  and so $\chi_A(\mathcal{Z}') = \chi_A(\mathcal{Z}'')$.

- Alternatively, one can show that

  $$0 = d_{\mathrm{GPr}}(\mathcal{X}, \mathcal{X}) = d_{\mathrm{GPr}}(\mathcal{Y} \boxplus \mathcal{Z}', \mathcal{Y} \boxplus \mathcal{Z}'') = d_{\mathrm{GPr}}(\mathcal{Z}', \mathcal{Z}'').$$

Cancellativity

- The commutative semigroup $(\mathbb{M}, \boxplus)$ is cancellative; that is, if $\mathcal{X}, \mathcal{Y}, \mathcal{Z}', \mathcal{Z}'' \in \mathbb{M}$ satisfy
$$\mathcal{X} = \mathcal{Y} \boxplus \mathcal{Z}'$$
and
$$\mathcal{X} = \mathcal{Y} \boxplus \mathcal{Z}'',$$
then
$$\mathcal{Z}' = \mathcal{Z}''.$$
- This is because for all $A \in \mathbb{A}$
$$\chi_A(\mathcal{Y})\chi_A(\mathcal{Z}') = \chi_A(X) = \chi_A(\mathcal{Y})\chi_A(\mathcal{Z}''),$$
and so $\chi_A(\mathcal{Z}') = \chi_A(\mathcal{Z}'')$.
- Alternatively, one can show that
$$0 = d_{\mathrm{GPr}}(\mathcal{X}, \mathcal{X}) = d_{\mathrm{GPr}}(\mathcal{Y} \boxplus \mathcal{Z}', \mathcal{Y} \boxplus \mathcal{Z}'') = d_{\mathrm{GPr}}(\mathcal{Z}', \mathcal{Z}'').$$

Cancellativity

- The commutative semigroup $(\mathbb{M}, \boxplus)$ is cancellative; that is, if $\mathcal{X}, \mathcal{Y}, \mathcal{Z}', \mathcal{Z}'' \in \mathbb{M}$ satisfy
$$\mathcal{X} = \mathcal{Y} \boxplus \mathcal{Z}'$$
and
$$\mathcal{X} = \mathcal{Y} \boxplus \mathcal{Z}'',$$
then
$$\mathcal{Z}' = \mathcal{Z}''.$$
- This is because for all $A \in \mathbb{A}$
$$\chi_A(\mathcal{Y})\chi_A(\mathcal{Z}') = \chi_A(X) = \chi_A(\mathcal{Y})\chi_A(\mathcal{Z}''),$$
and so $\chi_A(\mathcal{Z}') = \chi_A(\mathcal{Z}'')$.
- Alternatively, one can show that
$$0 = d_{\mathrm{GPr}}(\mathcal{X}, \mathcal{X}) = d_{\mathrm{GPr}}(\mathcal{Y} \boxplus \mathcal{Z}', \mathcal{Y} \boxplus \mathcal{Z}'') = d_{\mathrm{GPr}}(\mathcal{Z}', \mathcal{Z}'').$$

- Put $D_A(\mathcal{X}) := -\log \chi_A(\mathcal{X}) \geqslant 0$ and

$$D(\mathcal{X}) := D_1(\mathcal{X}) = -\log \chi_1(\mathcal{X}) = -\log \int_{X^2} \exp\left(-r_X(x_1, x_2)\right) \mu_X^{\otimes 2}(dx).$$

- Put

$$R(\mathcal{X}) := \int_{X^2} (r_X(x_1, x_2) \wedge 1) \, \mu_X^{\otimes 2}(dx).$$

- For suitable constants, $aD(\mathcal{X}) \leqslant D_A(\mathcal{X}) \leqslant bD(\mathcal{X})$.
- $\frac{1}{4} R(\mathcal{X}) \leqslant d_{\mathrm{GPr}}(\mathcal{X}, \mathcal{E}) \leqslant \sqrt{R(\mathcal{X})}$.
- For suitable constants, $\alpha(D(\mathcal{X}) \wedge 1) \leqslant R(\mathcal{X}) \leqslant \beta(D(\mathcal{X}) \wedge 1)$.

- Put $D_A(\mathcal{X}) := -\log \chi_A(\mathcal{X}) \geqslant 0$ and

$$D(\mathcal{X}) := D_1(\mathcal{X}) = -\log \chi_1(\mathcal{X}) = -\log \int_{X^2} \exp\left(-r_X(x_1, x_2)\right) \mu_X^{\otimes 2}(dx).$$

- Put

$$R(\mathcal{X}) := \int_{X^2} (r_X(x_1, x_2) \wedge 1) \, \mu_X^{\otimes 2}(dx).$$

- For suitable constants, $aD(\mathcal{X}) \leqslant D_A(\mathcal{X}) \leqslant bD(\mathcal{X})$.
- $\frac{1}{4} R(\mathcal{X}) \leqslant d_{\mathrm{GPr}}(\mathcal{X}, \mathcal{E}) \leqslant \sqrt{R(\mathcal{X})}$.
- For suitable constants, $\alpha(D(\mathcal{X}) \wedge 1) \leqslant R(\mathcal{X}) \leqslant \beta(D(\mathcal{X}) \wedge 1)$.

- Put $D_A(\mathcal{X}) := -\log \chi_A(\mathcal{X}) \geqslant 0$ and

$$D(\mathcal{X}) := D_1(\mathcal{X}) = -\log \chi_1(\mathcal{X}) = -\log \int_{X^2} \exp\left(-r_X(x_1, x_2)\right) \mu_X^{\otimes 2}(dx).$$

- Put

$$R(\mathcal{X}) := \int_{X^2} (r_X(x_1, x_2) \wedge 1) \, \mu_X^{\otimes 2}(dx).$$

- For suitable constants, $aD(\mathcal{X}) \leqslant D_A(\mathcal{X}) \leqslant bD(\mathcal{X})$.
- $\frac{1}{4} R(\mathcal{X}) \leqslant d_{\mathrm{GPr}}(\mathcal{X}, \mathcal{E}) \leqslant \sqrt{R(\mathcal{X})}$.
- For suitable constants, $\alpha(D(\mathcal{X}) \wedge 1) \leqslant R(\mathcal{X}) \leqslant \beta(D(\mathcal{X}) \wedge 1)$.

Measuring the size of a metric measure space

- Put $D_A(\mathcal{X}) := -\log \chi_A(\mathcal{X}) \geqslant 0$ and

$$D(\mathcal{X}) := D_1(\mathcal{X}) = -\log \chi_1(\mathcal{X}) = -\log \int_{X^2} \exp\left(-r_X(x_1, x_2)\right) \mu_X^{\otimes 2}(dx).$$

- Put

$$R(\mathcal{X}) := \int_{X^2} (r_X(x_1, x_2) \wedge 1)\, \mu_X^{\otimes 2}(dx).$$

- For suitable constants, $aD(\mathcal{X}) \leqslant D_A(\mathcal{X}) \leqslant bD(\mathcal{X})$.
- $\frac{1}{4} R(\mathcal{X}) \leqslant d_{\mathrm{GPr}}(\mathcal{X}, \mathcal{E}) \leqslant \sqrt{R(\mathcal{X})}$.
- For suitable constants, $\alpha(D(\mathcal{X}) \wedge 1) \leqslant R(\mathcal{X}) \leqslant \beta(D(\mathcal{X}) \wedge 1)$.

Convergence of sums

- Suppose that $(\mathcal{X}_n)_{n\in\mathbb{N}}$ is a sequence such that $\lim_{n\to\infty} \mathcal{X}_0 \boxplus \cdots \boxplus \mathcal{X}_n = \mathcal{Y}$ for some $\mathcal{Y} \in \mathbb{M}$. If $(\mathcal{X}'_n)_{n\in\mathbb{N}}$ is a sequence that is obtained by re-ordering the sequence $(\mathcal{X}_n)_{n\in\mathbb{N}}$, then $\lim_{n\to\infty} \mathcal{X}'_0 \boxplus \cdots \boxplus \mathcal{X}'_n = \mathcal{Y}$ also.
- The limit $\lim_{n\to\infty} \mathcal{X}_0 \boxplus \cdots \boxplus \mathcal{X}_n$ exists if and only if $\sum_n D(\mathcal{X}_n) < \infty$ (equivalently, $\sum_n R(\mathcal{X}_n) < \infty$).
- So, we can make sense of $\boxplus_{s\in S} \mathcal{X}_s$ for any family $(\mathcal{X})_{s\in S}$ without specifying the "order of summation".

Convergence of sums

- Suppose that $(\mathcal{X}_n)_{n \in \mathbb{N}}$ is a sequence such that $\lim_{n \to \infty} \mathcal{X}_0 \boxplus \cdots \boxplus \mathcal{X}_n = \mathcal{Y}$ for some $\mathcal{Y} \in \mathbb{M}$. If $(\mathcal{X}'_n)_{n \in \mathbb{N}}$ is a sequence that is obtained by re-ordering the sequence $(\mathcal{X}_n)_{n \in \mathbb{N}}$, then $\lim_{n \to \infty} \mathcal{X}'_0 \boxplus \cdots \boxplus \mathcal{X}'_n = \mathcal{Y}$ also.
- The limit $\lim_{n \to \infty} \mathcal{X}_0 \boxplus \cdots \boxplus \mathcal{X}_n$ exists if and only if $\sum_n D(\mathcal{X}_n) < \infty$ (equivalently, $\sum_n R(\mathcal{X}_n) < \infty$).
- So, we can make sense of $\boxplus_{s \in S} \mathcal{X}_s$ for any family $(\mathcal{X})_{s \in S}$ without specifying the "order of summation".

Convergence of sums

- Suppose that $(\mathcal{X}_n)_{n\in\mathbb{N}}$ is a sequence such that $\lim_{n\to\infty} \mathcal{X}_0 \boxplus \cdots \boxplus \mathcal{X}_n = \mathcal{Y}$ for some $\mathcal{Y} \in \mathbb{M}$. If $(\mathcal{X}'_n)_{n\in\mathbb{N}}$ is a sequence that is obtained by re-ordering the sequence $(\mathcal{X}_n)_{n\in\mathbb{N}}$, then $\lim_{n\to\infty} \mathcal{X}'_0 \boxplus \cdots \boxplus \mathcal{X}'_n = \mathcal{Y}$ also.
- The limit $\lim_{n\to\infty} \mathcal{X}_0 \boxplus \cdots \boxplus \mathcal{X}_n$ exists if and only if $\sum_n D(\mathcal{X}_n) < \infty$ (equivalently, $\sum_n R(\mathcal{X}_n) < \infty$).
- So, we can make sense of $\boxplus_{s\in S} \mathcal{X}_s$ for any family $(\mathcal{X})_{s\in S}$ without specifying the "order of summation".

Irreducible elements

- An element $\mathcal{X} \in \mathbb{M}$ is irreducible if $\mathcal{X} \neq \mathcal{E}$ and $\mathcal{Y} \leqslant \mathcal{X}$ for $\mathcal{Y} \in \mathbb{M}$ implies that $\mathcal{Y}$ is either $\mathcal{E}$ or $\mathcal{X}$.
- It is not clear *a priori* that there are irreducible elements. For example, the semigroup $\mathbb{R}_+$ with the usual addition operation has no irreducible elements in the analogous sense.
- If $\mathcal{X} \in \mathbb{M} \backslash \{\mathcal{E}\}$, then there is an irreducible element $\mathcal{Y} \in \mathbb{M}$ with $\mathcal{Y} \leqslant \mathcal{X}$ – this seems to be not at all obvious and to rely on some nontrivial stochastic analysis.
- In particular, if $\mathcal{X} \in \mathbb{M} \backslash \{\mathcal{E}\}$, then it is not possible that for all $n \in \mathbb{N}$ there exists $\mathcal{X}_n$ such that $\mathcal{X} = \mathcal{X}_n \boxplus \cdots \boxplus \mathcal{X}_n$ (n terms).

Irreducible elements

- An element $\mathcal{X} \in \mathbb{M}$ is irreducible if $\mathcal{X} \neq \mathcal{E}$ and $\mathcal{Y} \leqslant \mathcal{X}$ for $\mathcal{Y} \in \mathbb{M}$ implies that $\mathcal{Y}$ is either $\mathcal{E}$ or $\mathcal{X}$.
- It is not clear *a priori* that there are irreducible elements. For example, the semigroup $\mathbb{R}_+$ with the usual addition operation has no irreducible elements in the analogous sense.
- If $\mathcal{X} \in \mathbb{M} \backslash \{\mathcal{E}\}$, then there is an irreducible element $\mathcal{Y} \in \mathbb{M}$ with $\mathcal{Y} \leqslant \mathcal{X}$ – this seems to be not at all obvious and to rely on some nontrivial stochastic analysis.
- In particular, if $\mathcal{X} \in \mathbb{M} \backslash \{\mathcal{E}\}$, then it is not possible that for all $n \in \mathbb{N}$ there exists $\mathcal{X}_n$ such that $\mathcal{X} = \mathcal{X}_n \boxplus \cdots \boxplus \mathcal{X}_n$ (n terms).

Irreducible elements

- An element $\mathcal{X} \in \mathbb{M}$ is irreducible if $\mathcal{X} \neq \mathcal{E}$ and $\mathcal{Y} \leqslant \mathcal{X}$ for $\mathcal{Y} \in \mathbb{M}$ implies that $\mathcal{Y}$ is either $\mathcal{E}$ or $\mathcal{X}$.
- It is not clear *a priori* that there are irreducible elements. For example, the semigroup $\mathbb{R}_+$ with the usual addition operation has no irreducible elements in the analogous sense.
- If $\mathcal{X} \in \mathbb{M} \backslash \{\mathcal{E}\}$, then there is an irreducible element $\mathcal{Y} \in \mathbb{M}$ with $\mathcal{Y} \leqslant \mathcal{X}$ – this seems to be not at all obvious and to rely on some nontrivial stochastic analysis.
- In particular, if $\mathcal{X} \in \mathbb{M} \backslash \{\mathcal{E}\}$, then it is not possible that for all $n \in \mathbb{N}$ there exists $\mathcal{X}_n$ such that $\mathcal{X} = \mathcal{X}_n \boxplus \cdots \boxplus \mathcal{X}_n$ (n terms).

Irreducible elements

- An element $\mathcal{X} \in \mathbb{M}$ is irreducible if $\mathcal{X} \neq \mathcal{E}$ and $\mathcal{Y} \leqslant \mathcal{X}$ for $\mathcal{Y} \in \mathbb{M}$ implies that $\mathcal{Y}$ is either $\mathcal{E}$ or $\mathcal{X}$.
- It is not clear *a priori* that there are irreducible elements. For example, the semigroup $\mathbb{R}_+$ with the usual addition operation has no irreducible elements in the analogous sense.
- If $\mathcal{X} \in \mathbb{M} \backslash \{\mathcal{E}\}$, then there is an irreducible element $\mathcal{Y} \in \mathbb{M}$ with $\mathcal{Y} \leqslant \mathcal{X}$ – this seems to be not at all obvious and to rely on some nontrivial stochastic analysis.
- In particular, if $\mathcal{X} \in \mathbb{M} \backslash \{\mathcal{E}\}$, then it is not possible that for all $n \in \mathbb{N}$ there exists $\mathcal{X}_n$ such that $\mathcal{X} = \mathcal{X}_n \boxplus \cdots \boxplus \mathcal{X}_n$ (n terms).

Concrete examples of irreducible elements: totally geodesic spaces

- A metric space (W, r_W) is totally geodesic if any two points of W are joined by a unique geodesic segment.
- Any nontrivial closed subset X of a totally geodesic, complete, separable metric space (W, r_W) is irreducible, no matter what measure it is equipped with.
- Why? If (X, r_W) is isometric to $(Y \times Z, r_Y \oplus r_Z)$ for nontrivial Y and Z, then there will be four distinct points a, b, c, d in X that are isometric images of points of the form (y', z'), (y'', z'), (y', z''), (y'', z'') in $Y \times Z$. Thus,

$$\begin{aligned} r_W(a,b) &= r_W(c,d), & r_W(a,c) &= r_W(b,d), \\ r_W(a,d) &= r_W(a,b) + r_W(b,d), & r_W(a,d) &= r_W(a,c) + r_W(c,d), \\ r_W(b,c) &= r_W(a,b) + r_W(c,a), & r_W(b,c) &= r_W(b,d) + r_W(c,d). \end{aligned}$$

It follows from the third and fourth equalities that b and c are on the geodesic segment between a and d. We may therefore suppose that (W, r_W) is a closed subinterval of $\mathbb{R}$ and, without loss of generality, that $a < b < c < d$. The fifth and sixth equalities are then impossible.

Concrete examples of irreducible elements: totally geodesic spaces

- A metric space (W, r_W) is totally geodesic if any two points of W are joined by a unique geodesic segment.
- Any nontrivial closed subset X of a totally geodesic, complete, separable metric space (W, r_W) is irreducible, no matter what measure it is equipped with.
- Why? If (X, r_W) is isometric to $(Y \times Z, r_Y \oplus r_Z)$ for nontrivial Y and Z, then there will be four distinct points a, b, c, d in X that are isometric images of points of the form (y', z'), (y'', z'), (y', z''), (y'', z'') in $Y \times Z$. Thus,

$$\begin{aligned} r_W(a,b) &= r_W(c,d), & r_W(a,c) &= r_W(b,d), \\ r_W(a,d) &= r_W(a,b) + r_W(b,d), & r_W(a,d) &= r_W(a,c) + r_W(c,d), \\ r_W(b,c) &= r_W(a,b) + r_W(c,a), & r_W(b,c) &= r_W(b,d) + r_W(c,d). \end{aligned}$$

  It follows from the third and fourth equalities that b and c are on the geodesic segment between a and d. We may therefore suppose that (W, r_W) is a closed subinterval of $\mathbb{R}$ and, without loss of generality, that $a < b < c < d$. The fifth and sixth equalities are then impossible.

Concrete examples of irreducible elements: totally geodesic spaces

- A metric space (W, r_W) is totally geodesic if any two points of W are joined by a unique geodesic segment.
- Any nontrivial closed subset X of a totally geodesic, complete, separable metric space (W, r_W) is irreducible, no matter what measure it is equipped with.
- Why? If (X, r_W) is isometric to $(Y \times Z, r_Y \oplus r_Z)$ for nontrivial Y and Z, then there will be four distinct points a, b, c, d in X that are isometric images of points of the form (y', z'), (y'', z'), (y', z''), (y'', z'') in $Y \times Z$. Thus,

$$\begin{aligned} r_W(a,b) &= r_W(c,d), & r_W(a,c) &= r_W(b,d), \\ r_W(a,d) &= r_W(a,b) + r_W(b,d), & r_W(a,d) &= r_W(a,c) + r_W(c,d), \\ r_W(b,c) &= r_W(a,b) + r_W(c,a), & r_W(b,c) &= r_W(b,d) + r_W(c,d). \end{aligned}$$

  It follows from the third and fourth equalities that b and c are on the geodesic segment between a and d. We may therefore suppose that (W, r_W) is a closed subinterval of $\mathbb{R}$ and, without loss of generality, that $a < b < c < d$. The fifth and sixth equalities are then impossible.

Examples of totally geodesic spaces

- A Banach space $(X, \| \|)$ is totally geodesic if and only if it is strictly convex; that is, $x \neq y$ and $\|x'\| = \|x''\| = 1$ imply that $\|ax' + (1-a)x''\| < 1$ for all $0 < a < 1$.
- Strict convexity of $(X, \| \|)$ is implied by uniform convexity; that is, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|x'\| = \|x''\| = 1$ and $\|x' - x''\| \geqslant \varepsilon$ imply $\|\frac{x'+x''}{2}\| \leqslant 1 - \delta$.
- Any Hilbert space and the Banach spaces $L^p(S, \mathcal{S}, \lambda)$, $1 < p < \infty$, where λ is a σ-finite measure, are uniformly convex.

Examples of totally geodesic spaces

- A Banach space $(X, \| \, \|)$ is totally geodesic if and only if it is strictly convex; that is, $x \neq y$ and $\|x'\| = \|x''\| = 1$ imply that $\|ax' + (1-a)x''\| < 1$ for all $0 < a < 1$.
- Strict convexity of $(X, \| \, \|)$ is implied by uniform convexity; that is, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|x'\| = \|x''\| = 1$ and $\|x' - x''\| \geqslant \varepsilon$ imply $\|\frac{x'+x''}{2}\| \leqslant 1 - \delta$.
- Any Hilbert space and the Banach spaces $L^p(S, \mathcal{S}, \lambda)$, $1 < p < \infty$, where λ is a σ-finite measure, are uniformly convex.

Examples of totally geodesic spaces

- A Banach space $(X, \| \|)$ is totally geodesic if and only if it is strictly convex; that is, $x \neq y$ and $\|x'\| = \|x''\| = 1$ imply that $\|ax' + (1-a)x''\| < 1$ for all $0 < a < 1$.
- Strict convexity of $(X, \| \|)$ is implied by uniform convexity; that is, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|x'\| = \|x''\| = 1$ and $\|x' - x''\| \geqslant \varepsilon$ imply $\|\frac{x'+x''}{2}\| \leqslant 1 - \delta$.
- Any Hilbert space and the Banach spaces $L^p(S, \mathcal{S}, \lambda)$, $1 < p < \infty$, where λ is a σ-finite measure, are uniformly convex.

How many irreducible elements are there?

- The set of irreducible elements is a dense G_δ subset (i.e. countable intersection of open sets) of $\mathbb{M}$.
- In particular, the set of irreducible elements is uncountable.

How many irreducible elements are there?

- The set of irreducible elements is a dense G_δ subset (i.e. countable intersection of open sets) of $\mathbb{M}$.
- In particular, the set of irreducible elements is uncountable.

Prime elements

- An element $\mathcal{X} \in \mathbb{M}$ is prime if $\mathcal{X} \neq \mathcal{E}$ and $\mathcal{X} \leqslant \mathcal{Y} \boxplus \mathcal{Z}$ for $\mathcal{Y}, \mathcal{Z} \in \mathbb{M}$ implies that $\mathcal{X} \leqslant \mathcal{Y}$ or $\mathcal{X} \leqslant \mathcal{Z}$.
- Prime elements are clearly irreducible, but the converse is not *a priori* true. There are commutative, cancellative semigroups where the analogue of the converse is false.
- All irreducible elements of $\mathbb{M}$ are prime.
- The analogous result for the integers is the key to proving the fundamental theorem of arithmetic, and the usual proof uses Euclid's algorithm.

Prime elements

- An element $\mathcal{X} \in \mathbb{M}$ is prime if $\mathcal{X} \neq \mathcal{E}$ and $\mathcal{X} \leqslant \mathcal{Y} \boxplus \mathcal{Z}$ for $\mathcal{Y}, \mathcal{Z} \in \mathbb{M}$ implies that $\mathcal{X} \leqslant \mathcal{Y}$ or $\mathcal{X} \leqslant \mathcal{Z}$.
- Prime elements are clearly irreducible, but the converse is not *a priori* true. There are commutative, cancellative semigroups where the analogue of the converse is false.
- All irreducible elements of $\mathbb{M}$ are prime.
- The analogous result for the integers is the key to proving the fundamental theorem of arithmetic, and the usual proof uses Euclid's algorithm.

Prime elements

- An element $\mathcal{X} \in \mathbb{M}$ is prime if $\mathcal{X} \neq \mathcal{E}$ and $\mathcal{X} \leqslant \mathcal{Y} \boxplus \mathcal{Z}$ for $\mathcal{Y}, \mathcal{Z} \in \mathbb{M}$ implies that $\mathcal{X} \leqslant \mathcal{Y}$ or $\mathcal{X} \leqslant \mathcal{Z}$.
- Prime elements are clearly irreducible, but the converse is not *a priori* true. There are commutative, cancellative semigroups where the analogue of the converse is false.
- All irreducible elements of $\mathbb{M}$ are prime.
- The analogous result for the integers is the key to proving the fundamental theorem of arithmetic, and the usual proof uses Euclid's algorithm.

Prime elements

- An element $\mathcal{X} \in \mathbb{M}$ is prime if $\mathcal{X} \neq \mathcal{E}$ and $\mathcal{X} \leqslant \mathcal{Y} \boxplus \mathcal{Z}$ for $\mathcal{Y}, \mathcal{Z} \in \mathbb{M}$ implies that $\mathcal{X} \leqslant \mathcal{Y}$ or $\mathcal{X} \leqslant \mathcal{Z}$.
- Prime elements are clearly irreducible, but the converse is not *a priori* true. There are commutative, cancellative semigroups where the analogue of the converse is false.
- All irreducible elements of $\mathbb{M}$ are prime.
- The analogous result for the integers is the key to proving the fundamental theorem of arithmetic, and the usual proof uses Euclid's algorithm.

Prime factorization – the "fundamental theorem of arithmetic"

- Given any $\mathcal{X} \in \mathbb{M} \backslash \{\mathcal{E}\}$, there is either a finite sequence $(\mathcal{X}_n)_{n=0}^N$ or an infinite sequence $(\mathcal{X}_n)_{n=0}^\infty$ of irreducible elements of $\mathbb{M}$ such that $\mathcal{X} = \mathcal{X}_0 \boxplus \cdots \boxplus \mathcal{X}_N$ in the first case and $\mathcal{X} = \lim_{n\to\infty} \mathcal{X}_0 \boxplus \cdots \boxplus \mathcal{X}_n$ in the second.
- The sequence is unique up to the order of its terms.
- Each irreducible element appears a finite number of times, so the representation is specified by the irreducible elements that appear and their finite multiplicities.
- It follows that $(\mathbb{M}, \leqslant)$ is a distributive lattice: there is an analogue of the greatest common divisor (meet) and the least common multiple (join).

Prime factorization – the "fundamental theorem of arithmetic"

- Given any $\mathcal{X} \in \mathbb{M} \backslash \{\mathcal{E}\}$, there is either a finite sequence $(\mathcal{X}_n)_{n=0}^N$ or an infinite sequence $(\mathcal{X}_n)_{n=0}^\infty$ of irreducible elements of $\mathbb{M}$ such that $\mathcal{X} = \mathcal{X}_0 \boxplus \cdots \boxplus \mathcal{X}_N$ in the first case and $\mathcal{X} = \lim_{n \to \infty} \mathcal{X}_0 \boxplus \cdots \boxplus \mathcal{X}_n$ in the second.
- The sequence is unique up to the order of its terms.
- Each irreducible element appears a finite number of times, so the representation is specified by the irreducible elements that appear and their finite multiplicities.
- It follows that $(\mathbb{M}, \leq)$ is a distributive lattice: there is an analogue of the greatest common divisor (meet) and the least common multiple (join).

Prime factorization – the "fundamental theorem of arithmetic"

- Given any $\mathcal{X} \in \mathbb{M}\backslash\{\mathcal{E}\}$, there is either a finite sequence $(\mathcal{X}_n)_{n=0}^N$ or an infinite sequence $(\mathcal{X}_n)_{n=0}^\infty$ of irreducible elements of $\mathbb{M}$ such that $\mathcal{X} = \mathcal{X}_0 \boxplus \cdots \boxplus \mathcal{X}_N$ in the first case and $\mathcal{X} = \lim_{n\to\infty} \mathcal{X}_0 \boxplus \cdots \boxplus \mathcal{X}_n$ in the second.
- The sequence is unique up to the order of its terms.
- Each irreducible element appears a finite number of times, so the representation is specified by the irreducible elements that appear and their finite multiplicities.
- It follows that $(\mathbb{M}, \leq)$ is a distributive lattice: there is an analogue of the greatest common divisor (meet) and the least common multiple (join).

Prime factorization – the "fundamental theorem of arithmetic"

- Given any $\mathcal{X} \in \mathbb{M} \backslash \{\mathcal{E}\}$, there is either a finite sequence $(\mathcal{X}_n)_{n=0}^{N}$ or an infinite sequence $(\mathcal{X}_n)_{n=0}^{\infty}$ of irreducible elements of $\mathbb{M}$ such that $\mathcal{X} = \mathcal{X}_0 \boxplus \cdots \boxplus \mathcal{X}_N$ in the first case and $\mathcal{X} = \lim_{n\to\infty} \mathcal{X}_0 \boxplus \cdots \boxplus \mathcal{X}_n$ in the second.
- The sequence is unique up to the order of its terms.
- Each irreducible element appears a finite number of times, so the representation is specified by the irreducible elements that appear and their finite multiplicities.
- It follows that $(\mathbb{M}, \leqslant)$ is a distributive lattice: there is an analogue of the greatest common divisor (meet) and the least common multiple (join).

- Existence of factorizations into irreducibles uses general results about Delphic semigroups from Kendall (1968), Davidson (1969).
- An ingredient for uniqueness is the existence of common refinements: If

$$\mathcal{X}_{0\bullet} \boxplus \mathcal{X}_{1\bullet} = \mathcal{X} = \mathcal{X}_{\bullet 0} \boxplus \mathcal{X}_{\bullet 1},$$

  then there exist $\mathcal{X}_{00}, \mathcal{X}_{01}, \mathcal{X}_{10}, \mathcal{X}_{11}$ such that

$$\mathcal{X}_{0\bullet} = \mathcal{X}_{00} \boxplus \mathcal{X}_{01}, \quad \mathcal{X}_{1\bullet} = \mathcal{X}_{10} \boxplus \mathcal{X}_{11}, \quad \mathcal{X}_{\bullet 0} = \mathcal{X}_{00} \boxplus \mathcal{X}_{10}, \quad \mathcal{X}_{\bullet 1} = \mathcal{X}_{01} \boxplus \mathcal{X}_{11}.$$

- The proof that common refinements exist uses some of the same ideas as Tardif's proof and the following elementary fact (where $\perp\!\!\!\perp$ denotes independence of random elements): Let $\xi_{00}, \xi_{01}, \xi_{10}, \xi_{11}$ be random elements of the respective complete separable metric spaces $X_{00}, X_{01}, X_{10}, X_{11}$. Suppose that

$$(\xi_{00}, \xi_{01}) \perp\!\!\!\perp (\xi_{10}, \xi_{11})$$

  and

$$(\xi_{00}, \xi_{10}) \perp\!\!\!\perp (\xi_{01}, \xi_{11}).$$

  Then,

$$\xi_{00} \perp\!\!\!\perp \xi_{01} \perp\!\!\!\perp \xi_{10} \perp\!\!\!\perp \xi_{11}.$$

A little about the proof

- Existence of factorizations into irreducibles uses general results about Delphic semigroups from Kendall (1968), Davidson (1969).
- An ingredient for uniqueness is the existence of common refinements: If

$$\mathcal{X}_{0\bullet} \boxplus \mathcal{X}_{1\bullet} = \mathcal{X} = \mathcal{X}_{\bullet 0} \boxplus \mathcal{X}_{\bullet 1},$$

  then there exist $\mathcal{X}_{00}, \mathcal{X}_{01}, \mathcal{X}_{10}, \mathcal{X}_{11}$ such that

$$\mathcal{X}_{0\bullet} = \mathcal{X}_{00} \boxplus \mathcal{X}_{01}, \quad \mathcal{X}_{1\bullet} = \mathcal{X}_{10} \boxplus \mathcal{X}_{11}, \quad \mathcal{X}_{\bullet 0} = \mathcal{X}_{00} \boxplus \mathcal{X}_{10}, \quad \mathcal{X}_{\bullet 1} = \mathcal{X}_{01} \boxplus \mathcal{X}_{11}.$$

- The proof that common refinements exist uses some of the same ideas as Tardif's proof and the following elementary fact (where $\perp\!\!\!\perp$ denotes independence of random elements): Let $\xi_{00}, \xi_{01}, \xi_{10}, \xi_{11}$ be random elements of the respective complete separable metric spaces $X_{00}, X_{01}, X_{10}, X_{11}$. Suppose that

$$(\xi_{00}, \xi_{01}) \perp\!\!\!\perp (\xi_{10}, \xi_{11})$$

  and

$$(\xi_{00}, \xi_{10}) \perp\!\!\!\perp (\xi_{01}, \xi_{11}).$$

  Then,

$$\xi_{00} \perp\!\!\!\perp \xi_{01} \perp\!\!\!\perp \xi_{10} \perp\!\!\!\perp \xi_{11}.$$

A little about the proof

- Existence of factorizations into irreducibles uses general results about Delphic semigroups from Kendall (1968), Davidson (1969).
- An ingredient for uniqueness is the existence of common refinements: If

$$\mathcal{X}_{0\bullet} \boxplus \mathcal{X}_{1\bullet} = \mathcal{X} = \mathcal{X}_{\bullet 0} \boxplus \mathcal{X}_{\bullet 1},$$

  then there exist $\mathcal{X}_{00}, \mathcal{X}_{01}, \mathcal{X}_{10}, \mathcal{X}_{11}$ such that

$$\mathcal{X}_{0\bullet} = \mathcal{X}_{00} \boxplus \mathcal{X}_{01}, \quad \mathcal{X}_{1\bullet} = \mathcal{X}_{10} \boxplus \mathcal{X}_{11}, \quad \mathcal{X}_{\bullet 0} = \mathcal{X}_{00} \boxplus \mathcal{X}_{10}, \quad \mathcal{X}_{\bullet 1} = \mathcal{X}_{01} \boxplus \mathcal{X}_{11}.$$

- The proof that common refinements exist uses some of the same ideas as Tardif's proof and the following elementary fact (where $\perp\!\!\!\perp$ denotes independence of random elements): Let $\xi_{00}, \xi_{01}, \xi_{10}, \xi_{11}$ be random elements of the respective complete separable metric spaces $X_{00}, X_{01}, X_{10}, X_{11}$. Suppose that

$$(\xi_{00}, \xi_{01}) \perp\!\!\!\perp (\xi_{10}, \xi_{11})$$

  and

$$(\xi_{00}, \xi_{10}) \perp\!\!\!\perp (\xi_{01}, \xi_{11}).$$

  Then,

$$\xi_{00} \perp\!\!\!\perp \xi_{01} \perp\!\!\!\perp \xi_{10} \perp\!\!\!\perp \xi_{11}.$$

- A random element $\mathbf{Y}$ of $\mathbb{M}$ is infinitely divisible if for each $n \in \mathbb{N}$ there are independent, identically distributed, random elements $\mathbf{Y}_{n1}, \ldots, \mathbf{Y}_{nn}$ such that $\mathbf{Y}$ has the same probability distribution as $\mathbf{Y}_{n1} \boxplus \cdots \boxplus \mathbf{Y}_{nn}$.
- An infinitely divisible random element $\mathcal{Y}$ has the same probability distribution as

$$\boxplus\{\mathcal{X} : (t, \mathcal{X}) \in \Pi\},$$

where Π is a Poisson random measure on $[0,1] \times (\mathbb{M} \setminus \{\mathcal{E}\})$ with intensity measure of the form $\lambda \otimes \nu$, where λ is Lebesgue measure and and ν is a σ-finite measure on $\mathbb{M} \setminus \{\mathcal{E}\}$ such that

$$\int D(\mathcal{X}) \wedge 1 \, \nu(d\mathcal{X}) < \infty.$$

- Conversely, any such measure ν corresponds to an infinitely divisible random element in this way.
- Constants are not infinitely divisible and there is no analogue of the Gaussian probability distribution.

- A random element $\mathbf{Y}$ of $\mathbb{M}$ is infinitely divisible if for each $n \in \mathbb{N}$ there are independent, identically distributed, random elements $\mathbf{Y}_{n1}, \ldots, \mathbf{Y}_{nn}$ such that $\mathbf{Y}$ has the same probability distribution as $\mathbf{Y}_{n1} \boxplus \cdots \boxplus \mathbf{Y}_{nn}$.
- An infinitely divisible random element $\mathcal{Y}$ has the same probability distribution as

$$\boxplus \{\mathcal{X} : (t, \mathcal{X}) \in \Pi\},$$

where Π is a Poisson random measure on $[0,1] \times (\mathbb{M} \backslash \{\mathcal{E}\})$ with intensity measure of the form $\lambda \otimes \nu$, where λ is Lebesgue measure and and ν is a σ-finite measure on $\mathbb{M} \backslash \{\mathcal{E}\}$ such that

$$\int D(\mathcal{X}) \wedge 1 \, \nu(d\mathcal{X}) < \infty.$$

- Conversely, any such measure ν corresponds to an infinitely divisible random element in this way.
- Constants are not infinitely divisible and there is no analogue of the Gaussian probability distribution.

- A random element $\mathbf{Y}$ of $\mathbb{M}$ is infinitely divisible if for each $n \in \mathbb{N}$ there are independent, identically distributed, random elements $\mathbf{Y}_{n1}, \ldots, \mathbf{Y}_{nn}$ such that $\mathbf{Y}$ has the same probability distribution as $\mathbf{Y}_{n1} \boxplus \cdots \boxplus \mathbf{Y}_{nn}$.
- An infinitely divisible random element $\mathcal{Y}$ has the same probability distribution as

$$\boxplus \{\mathcal{X} : (t, \mathcal{X}) \in \Pi\},$$

where Π is a Poisson random measure on $[0,1] \times (\mathbb{M} \backslash \{\mathcal{E}\})$ with intensity measure of the form $\lambda \otimes \nu$, where λ is Lebesgue measure and and ν is a σ-finite measure on $\mathbb{M} \backslash \{\mathcal{E}\}$ such that

$$\int D(\mathcal{X}) \wedge 1 \, \nu(d\mathcal{X}) < \infty.$$

- Conversely, any such measure ν corresponds to an infinitely divisible random element in this way.
- Constants are not infinitely divisible and there is no analogue of the Gaussian probability distribution.

- A random element $\mathbf{Y}$ of $\mathbb{M}$ is infinitely divisible if for each $n \in \mathbb{N}$ there are independent, identically distributed, random elements $\mathbf{Y}_{n1}, \ldots, \mathbf{Y}_{nn}$ such that $\mathbf{Y}$ has the same probability distribution as $\mathbf{Y}_{n1} \boxplus \cdots \boxplus \mathbf{Y}_{nn}$.
- An infinitely divisible random element $\mathcal{Y}$ has the same probability distribution as

$$\boxplus \{\mathcal{X} : (t, \mathcal{X}) \in \Pi\},$$

  where Π is a Poisson random measure on $[0,1] \times (\mathbb{M} \backslash \{\mathcal{E}\})$ with intensity measure of the form $\lambda \otimes \nu$, where λ is Lebesgue measure and and ν is a σ-finite measure on $\mathbb{M} \backslash \{\mathcal{E}\}$ such that

$$\int D(\mathcal{X}) \wedge 1 \, \nu(d\mathcal{X}) < \infty.$$

- Conversely, any such measure ν corresponds to an infinitely divisible random element in this way.
- Constants are not infinitely divisible and there is no analogue of the Gaussian probability distribution.

Scaling

- Given $\mathcal{X} \in \mathbb{M}$ and $a > 0$, set $a\mathcal{X} := (X, ar_X, \mu_X) \in \mathbb{M}$.
- This scaling operation operation is continuous and satisfies

$$a(\mathcal{X} \boxplus \mathcal{Y}) = (a\mathcal{X}) \boxplus (a\mathcal{Y}).$$

- If

$$(a\mathcal{X}) \boxplus (b\mathcal{X}) = c\mathcal{X}$$

for some $\mathcal{X} \in \mathbb{M}$ and $a, b, c > 0$, then $\mathcal{X} = \mathcal{E}$, so the second distributivity law certainly does not hold.

Scaling

- Given $\mathcal{X} \in \mathbb{M}$ and $a > 0$, set $a\mathcal{X} := (X, ar_X, \mu_X) \in \mathbb{M}$.
- This scaling operation operation is continuous and satisfies

$$a(\mathcal{X} \boxplus \mathcal{Y}) = (a\mathcal{X}) \boxplus (a\mathcal{Y}).$$

- If

$$(a\mathcal{X}) \boxplus (b\mathcal{X}) = c\mathcal{X}$$

for some $\mathcal{X} \in \mathbb{M}$ and $a, b, c > 0$, then $\mathcal{X} = \mathcal{E}$, so the second distributivity law certainly does not hold.

Scaling

- Given $\mathcal{X} \in \mathbb{M}$ and $a > 0$, set $a\mathcal{X} := (X, ar_X, \mu_X) \in \mathbb{M}$.
- This scaling operation operation is continuous and satisfies

$$a(\mathcal{X} \boxplus \mathcal{Y}) = (a\mathcal{X}) \boxplus (a\mathcal{Y}).$$

- If

$$(a\mathcal{X}) \boxplus (b\mathcal{X}) = c\mathcal{X}$$

for some $\mathcal{X} \in \mathbb{M}$ and $a, b, c > 0$, then $\mathcal{X} = \mathcal{E}$, so the second distributivity law certainly does not hold.

Stable random elements – “LePage” representations

- A $\mathbb{M}$-valued random element $\mathbf{Y}$ is stable with index $\alpha > 0$ if for any $a, b > 0$ the random element

$$(a+b)^{\frac{1}{\alpha}} \mathbf{Y}$$

has the same distribution as

$$a^{\frac{1}{\alpha}} \mathbf{Y}' \boxplus b^{\frac{1}{\alpha}} \mathbf{Y}''.$$

- A stable random element is necessarily infinitely divisible.
- The index must satisfy $0 < \alpha < 1$.
- An α-stable random element has the same distribution as

$$\boxplus_{n \in \mathbb{N}} \Gamma_n^{-\frac{1}{\alpha}} \mathbf{Z}_n,$$

where $(\Gamma_n)_{n \in \mathbb{N}}$ is the sequence of successive arrivals of a homogeneous unit intensity Poisson point process on $\mathbb{R}_+$ and $(\mathbf{Z}_n)_{n \in \mathbb{N}}$ is a sequence of i.i.d. random elements of $\mathbb{M}$.

- A $\mathbb{M}$-valued random element $\mathbf{Y}$ is stable with index $\alpha > 0$ if for any $a, b > 0$ the random element

$$(a+b)^{\frac{1}{\alpha}}\mathbf{Y}$$

has the same distribution as

$$a^{\frac{1}{\alpha}}\mathbf{Y}' \boxplus b^{\frac{1}{\alpha}}\mathbf{Y}''.$$

- A stable random element is necessarily infinitely divisible.
- The index must satisfy $0 < \alpha < 1$.
- An α-stable random element has the same distribution as

$$\boxplus_{n \in \mathbb{N}} \Gamma_n^{-\frac{1}{\alpha}} \mathbf{Z}_n,$$

where $(\Gamma_n)_{n\in\mathbb{N}}$ is the sequence of successive arrivals of a homogeneous unit intensity Poisson point process on $\mathbb{R}_+$ and $(\mathbf{Z}_n)_{n\in\mathbb{N}}$ is a sequence of i.i.d. random elements of $\mathbb{M}$.

Stable random elements – "LePage" representations

- A $\mathbb{M}$-valued random element $\mathbf{Y}$ is stable with index $\alpha > 0$ if for any $a, b > 0$ the random element

$$(a+b)^{\frac{1}{\alpha}}\mathbf{Y}$$

has the same distribution as

$$a^{\frac{1}{\alpha}}\mathbf{Y}' \boxplus b^{\frac{1}{\alpha}}\mathbf{Y}''.$$

- A stable random element is necessarily infinitely divisible.
- The index must satisfy $0 < \alpha < 1$.
- An α-stable random element has the same distribution as

$$\boxplus_{n \in \mathbb{N}} \Gamma_n^{-\frac{1}{\alpha}} \mathbf{Z}_n,$$

where $(\Gamma_n)_{n \in \mathbb{N}}$ is the sequence of successive arrivals of a homogeneous unit intensity Poisson point process on $\mathbb{R}_+$ and $(\mathbf{Z}_n)_{n \in \mathbb{N}}$ is a sequence of i.i.d. random elements of $\mathbb{M}$.

Stable random elements – "LePage" representations

- A $\mathbb{M}$-valued random element $\mathbf{Y}$ is stable with index $\alpha > 0$ if for any $a, b > 0$ the random element

$$(a+b)^{\frac{1}{\alpha}}\mathbf{Y}$$

has the same distribution as

$$a^{\frac{1}{\alpha}}\mathbf{Y}' \boxplus b^{\frac{1}{\alpha}}\mathbf{Y}''.$$

- A stable random element is necessarily infinitely divisible.
- The index must satisfy $0 < \alpha < 1$.
- An α-stable random element has the same distribution as

$$\boxplus_{n \in \mathbb{N}} \Gamma_n^{-\frac{1}{\alpha}} \mathbf{Z}_n,$$

where $(\Gamma_n)_{n\in\mathbb{N}}$ is the sequence of successive arrivals of a homogeneous unit intensity Poisson point process on $\mathbb{R}_+$ and $(\mathbf{Z}_n)_{n\in\mathbb{N}}$ is a sequence of i.i.d. random elements of $\mathbb{M}$.

- Cancellativity allows us to embed the semigroup $\mathbb{M}$ into a group – analogous to passing from $\mathbb{N}$ to $\mathbb{Z}$.
- Are there analogues of objects such as Gaussian random variables and Brownian motion on this group?
- What if we combine metrics via

$$((y', z'), (y'', z'')) \mapsto r_Y(y', y'') \vee r_Z(z', z'')$$

instead of

$$((y', z'), (y'', z'')) \mapsto r_Y(y', y'') + r_Z(z', z'')$$

(that is, "ℓ^∞" instead of "ℓ^1" – corresponds to the strong product of two graphs)?
- Rieffel has shown that one can obtain a quantum analogue of the space of compact metric spaces equipped with the Gromov–Hausdorff distance by considering C^*-algebras with properties that generalize those of the algebra of Lipschitz functions on a compact metric space.
- Is there a similar quantization for metric measure spaces? Ongoing work with Benson Au.

Future directions

- Cancellativity allows us to embed the semigroup $\mathbb{M}$ into a group – analogous to passing from $\mathbb{N}$ to $\mathbb{Z}$.
- Are there analogues of objects such as Gaussian random variables and Brownian motion on this group?
- What if we combine metrics via

$$((y', z'), (y'', z'')) \mapsto r_Y(y', y'') \vee r_Z(z', z'')$$

  instead of

$$((y', z'), (y'', z'')) \mapsto r_Y(y', y'') + r_Z(z', z'')$$

  (that is, "ℓ^∞" instead of "ℓ^1" – corresponds to the strong product of two graphs)?
- Rieffel has shown that one can obtain a quantum analogue of the space of compact metric spaces equipped with the Gromov–Hausdorff distance by considering C^*-algebras with properties that generalize those of the algebra of Lipschitz functions on a compact metric space.
- Is there a similar quantization for metric measure spaces? Ongoing work with Benson Au.

Future directions

- Cancellativity allows us to embed the semigroup $\mathbb{M}$ into a group – analogous to passing from $\mathbb{N}$ to $\mathbb{Z}$.
- Are there analogues of objects such as Gaussian random variables and Brownian motion on this group?
- What if we combine metrics via

$$((y', z'), (y'', z'')) \mapsto r_Y(y', y'') \vee r_Z(z', z'')$$

  instead of

$$((y', z'), (y'', z'')) \mapsto r_Y(y', y'') + r_Z(z', z'')$$

  (that is, "ℓ^∞" instead of "ℓ^1" – corresponds to the strong product of two graphs)?
- Rieffel has shown that one can obtain a quantum analogue of the space of compact metric spaces equipped with the Gromov–Hausdorff distance by considering C^*-algebras with properties that generalize those of the algebra of Lipschitz functions on a compact metric space.
- Is there a similar quantization for metric measure spaces? Ongoing work with Benson Au.

Future directions

- Cancellativity allows us to embed the semigroup $\mathbb{M}$ into a group – analogous to passing from $\mathbb{N}$ to $\mathbb{Z}$.
- Are there analogues of objects such as Gaussian random variables and Brownian motion on this group?
- What if we combine metrics via

$$((y', z'), (y'', z'')) \mapsto r_Y(y', y'') \vee r_Z(z', z'')$$

instead of

$$((y', z'), (y'', z'')) \mapsto r_Y(y', y'') + r_Z(z', z'')$$

(that is, "ℓ^∞" instead of "ℓ^1" – corresponds to the strong product of two graphs)?

- Rieffel has shown that one can obtain a quantum analogue of the space of compact metric spaces equipped with the Gromov–Hausdorff distance by considering C^*-algebras with properties that generalize those of the algebra of Lipschitz functions on a compact metric space.
- Is there a similar quantization for metric measure spaces? Ongoing work with Benson Au.

Future directions

- Cancellativity allows us to embed the semigroup $\mathbb{M}$ into a group – analogous to passing from $\mathbb{N}$ to $\mathbb{Z}$.
- Are there analogues of objects such as Gaussian random variables and Brownian motion on this group?
- What if we combine metrics via

$$((y', z'), (y'', z'')) \mapsto r_Y(y', y'') \vee r_Z(z', z'')$$

  instead of

$$((y', z'), (y'', z'')) \mapsto r_Y(y', y'') + r_Z(z', z'')$$

  (that is, "ℓ^∞" instead of "ℓ^1" – corresponds to the strong product of two graphs)?
- Rieffel has shown that one can obtain a quantum analogue of the space of compact metric spaces equipped with the Gromov–Hausdorff distance by considering C^*-algebras with properties that generalize those of the algebra of Lipschitz functions on a compact metric space.
- Is there a similar quantization for metric measure spaces? Ongoing work with Benson Au.