The exponential homomorphism in non-commutative probability

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Classical convolutions.

Additive:

$$\int_{\mathbb{R}} f(z)\, d(\mu_1 * \mu_2)(z) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x+y)\, d\mu_1(x)\, d\mu_2(y).$$

Multiplicative:

$$\int_{\mathbb{T}} f(z)\, d(\nu_1 \circledast \nu_2)(z) = \int_{\mathbb{T}} \int_{\mathbb{T}} f(zw)\, d\nu_1(z)\, d\nu_2(w).$$

The **wrapping map** $W : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{T})$ is

$$d(W(\mu))(e^{-ix}) = \sum_{n \in \mathbb{Z}} d\mu(x + 2\pi n).$$

Clearly

$$W(\mu_1 * \mu_2) = W(\mu_1) \circledast W(\mu_2).$$

Non-commutative independence.

Non-commutative convolutions: based on different independence rules.

Tensor/classical $\mathbb{E}[xyxy] = \mathbb{E}[x^2]\,\mathbb{E}[y^2]$.

Free $\mathbb{E}[xyxy] = \mathbb{E}[x^2]\,\mathbb{E}[y]^2 + \mathbb{E}[x]^2\,\mathbb{E}[y^2] - \mathbb{E}[x]^2\,\mathbb{E}[y]^2$.

Boolean $\mathbb{E}[xyxy] = \mathbb{E}[x]^2\,\mathbb{E}[y]^2$.

Monotone $\mathbb{E}[xyxy] = \mathbb{E}[x^2]\,\mathbb{E}[y]^2$.

Convolutions.

Additive convolutions: for measures $\mu_1, \mu_2 \in \mathcal{P}(\mathbb{R})$.

Classical $\mu_1 * \mu_2$, free $\mu_1 \boxplus \mu_2$, Boolean $\mu_1 \uplus \mu_2$, monotone $\mu_1 \rhd \mu_2$.

Multiplicative convolutions: for measures $\nu_1, \nu_2 \in \mathcal{P}(\mathbb{T})$.

Classical $\nu_1 \circledast \nu_2$, free $\nu_1 \boxtimes \nu_2$, Boolean $\nu_1 \utimes \nu_2$, monotone $\nu_1 \circlearrowright \nu_2$.

Transforms.

The F-transform:

$$\mu \in \mathcal{P}(\mathbb{R}), \quad G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-x}\, d\mu(x), \quad F_\mu(z) = \frac{1}{G_\mu(z)}.$$

$F_\mu : \mathbb{C}^+ \to \mathbb{C}^+$, $\lim_{y \uparrow \infty} \frac{\Im F_\mu(iy)}{iy} = 1$.

The η-transform:

$$\nu \in \mathcal{P}(\mathbb{T}), \quad \psi_\nu(z) = \int_{\mathbb{T}} \frac{z\zeta}{1-z\zeta}\, d\nu(\zeta), \quad \eta_\nu(z) = \frac{\psi_\nu(z)}{1+\psi_\nu(z)}.$$

$\eta_\nu : \mathbb{D} \to \mathbb{D}$, $\eta_\nu(0) = 0$.

Convolutions in non-commutative probability.

Additive convolutions:

Free $\mu \boxplus \nu$: $F^{-1}_{\mu \boxplus \nu}(z) - z = (F^{-1}_{\mu}(z) - z) + (F^{-1}_{\nu}(z) - z)$,

Boolean $\mu \uplus \nu$: $F_{\mu \uplus \nu}(z) - z = (F_{\mu}(z) - z) + (F_{\nu}(z) - z)$,

Monotone $\mu \rhd \nu$: $F_{\mu \rhd \nu}(z) = F_{\mu}(F_{\nu}(z))$.

Multiplicative convolutions:

Free $\mu \boxtimes \nu$: $\dfrac{\eta^{-1}_{\mu \boxtimes \nu}(z)}{z} = \dfrac{\eta^{-1}_{\mu}(z)}{z} \dfrac{\eta^{-1}_{\nu}(z)}{z}$,

Boolean $\mu \, \circledtimes \, \nu$: $\dfrac{\eta_{\mu \circledtimes \nu}(z)}{z} = \dfrac{\eta_{\mu}(z)}{z} \dfrac{\eta_{\nu}(z)}{z}$,

Monotone $\mu \circlearrowright \nu$: $\eta_{\mu \circlearrowright \nu}(z) = \eta_{\mu} \circ \eta_{\nu}(z)$.

Homomorphisms.

W is certainly not a homomorphism between free additive and multiplicative convolutions.

Example.

Let $\mu = \frac{1}{2}(\delta_{-2\pi} + \delta_{2\pi})$ be a Bernoulli distribution. Then $W(\mu) = \delta_1$. Also, it is well-known that $\mu \boxplus \mu$ is an arcsine distribution, while $\delta_1 \boxtimes \delta_1 = \delta_1$. Thus $W(\mu \boxplus \mu) \neq W(\mu) \boxtimes W(\mu)$.

Successful homomorphisms between $\boxplus$ and $\boxtimes$ on the level of power series: (Mastnak, Nica 2010), (Friedrich, McKay 2012, 2013).

A homomorphism between $\boxplus$ and $\boxtimes$ infinitely divisible distributions: (Cebron 2014).

Homomorphisms II.

Define an implicit relation between $\mu \in \mathcal{P}(\mathbb{R})$ and $\nu \in \mathcal{P}(\mathbb{T})$ by

$$\boxed{\exp(iF_\mu(z)) = \eta_\nu(e^{iz})}.$$

Then "obviously"

$$\mu_1 \boxplus \mu_2 \leftrightarrow \nu_1 \boxtimes \nu_2, \quad \mu_1 \uplus \mu_2 \leftrightarrow \nu_1 \utimes \nu_2, \quad \mu_1 \rhd \mu_2 \leftrightarrow \nu_1 \circlearrowright \nu_2.$$

In fact also

$$\mu^{\uplus t} \leftrightarrow \nu^{\utimes t}, \quad \mu^{\boxplus t} \leftrightarrow \nu^{\boxtimes t}, \quad \mu_1 \boxright \mu_2 \leftrightarrow \nu_1 \boxslash \nu_2.$$

(here $F_{\mu_1 \boxright \mu_2} = F_{\mu_2} \circ F_{\mu_1 \boxright \mu_2}$, $\eta_{\nu_1 \boxtimes \nu_2} = \eta_{\nu_2} \circ \eta_{\nu_1 \boxslash \nu_2}$).

$$\mu^{\rhd t} \leftrightarrow \nu^{\circlearrowright t}?$$

Identities.

Easily obtain multiplicative identities from additive ones, for example of

$$\mu = \mu^{\boxplus t} \rhd \mu^{\uplus (1-t)}$$

and

$$\mathbb{B}_t(\tau \boxplus \nu) = \tau \boxplus (\nu \boxplus \tau^{\boxplus t})$$

(particular case obtained in Zhong 2014).

$\mathbb{M}_t$ = multiplicative version of the Belinschi-Nica transformation $\mathbb{B}_t$. Use these to define multiplicative and additive free divisibility indicators.

Proposition.

For $\mu \in \mathcal{L}$, the additive divisibility indicator of μ is equal to the multiplicative divisibility indicator of $W(\mu)$.

Class $\mathcal{L}$.

$$\boxed{\exp(iF_\mu(z)) = \eta_\nu(e^{iz}).}$$

Domain:

$$\{\mu \in \mathcal{P}(\mathbb{R}) : F_\mu(z + 2\pi) = F_\mu(z) + 2\pi\} = \mathcal{L}.$$

Range:

$$\left\{\nu \in \mathcal{P}(\mathbb{T}) : \eta_\nu'(0) \neq 0, \text{ and } \eta_\nu(z) = 0 \Leftrightarrow z = 0\right\} = \mathcal{ID}_*^{\circledast}.$$

$$\mathcal{ID}_*^{\circledast} = \left\{\nu \in \mathcal{P}(\mathbb{T}) : \nu^{\circledast t} \text{ exists for } t \geqslant 0, \nu \neq \text{ Lebesgue}\right\}.$$

The wrapping homomorphism.

Theorem. (A, Arizmendi 2015)

When restricted to $\mathcal{L}$, the wrapping map W satisfies

$$\exp(iF_\mu(z)) = \eta_{W(\mu)}(e^{iz}).$$

Therefore this restriction is a homomorphism for all four additive convolutions, and has the additional properties mentioned above. The pre-image of each $\nu \in \mathcal{ID}_*^{\circledast}$ is an equivalence class modulo the relation $\mod \delta_{2\pi}$, where any of the four convolutions with $\delta_{2\pi}$ is used.

Proof of the Theorem.

Poisson summation.

Domain.

Proposition.

- $\mathcal{L}$ is closed under the three additive convolution operations $\uplus, \boxplus, \rhd$, under the subordination operation $\boxright$, under Boolean and free (whenever defined) additive convolution powers, and under the Belinschi-Nica transformation $\mathbb{B}_t$.
- If $\mu \in \mathcal{L} \cap \mathcal{ID}^{\rhd}$, then $\mu^{\rhd t} \in \mathcal{L}$ for all $t > 0$.
- All the elements in $\mathcal{L}$ which are not point masses are in the classical, Boolean, free, and monotone (strict) domains of attraction of the Cauchy law.
- The Bercovici-Pata bijections between $\mathcal{P} = \mathcal{ID}^{\uplus}$, $\mathcal{ID}^{\rhd}$, and $\mathcal{ID}^{\boxplus}$ restrict to bijections between $\mathcal{L}$, $\mathcal{L} \cap \mathcal{ID}^{\rhd}$ and $\mathcal{L} \cap \mathcal{ID}^{\boxplus}$.

Range.

Proposition.

- $\mathcal{ID}_*^{\circledast}$ is closed under the three multiplicative convolution operations $\circledast$, $\boxtimes$, $\circlearrowright$, under the subordination operation $\boxslash$, and under Boolean and free (whenever defined) multiplicative convolution powers.
- $\mathcal{ID}_*^{\circledast}$ contains $\mathcal{ID}_*^{\boxtimes}$ and $\mathcal{ID}_*^{\circlearrowright}$.
- If $\nu \in \mathcal{ID}_*^{\boxtimes}$, then every element of $W^{-1}(\nu)$ is in $\mathcal{ID}^{\boxplus} \cap \mathcal{L}$.
- If $\nu \in \mathcal{ID}_*^{\circlearrowright}$, then there is $\mu \in \mathcal{ID}_*^{\rhd} \cap \mathcal{L}$ such that $W(\mu) = \nu$.

Example of $\mu \in \mathcal{L}$ I.

Cauchy distribution

$$\mu = \frac{1}{\pi} \frac{a}{(x-b)^2 + a^2} \, dx.$$

Wrapped Cauchy distribution

$$W(\mu) = \frac{1}{2\pi} \frac{1 - e^{-2a}}{1 + e^{-2a} - 2e^{-a} \cos(\theta - b)} \, d\theta.$$

Example of $\mu \in \mathcal{L}$ II.

Pre-image of the multiplicative Boolean Gaussian.

$$\mu = \sum \alpha_k \delta_{x_k},$$

where

$$x_k = \cot \frac{x_k}{2}, \quad x_i \in \left(-\frac{\pi}{2} + \pi k, \frac{\pi}{2} + \pi k\right)$$

and

$$\alpha_k = \frac{1}{\frac{3}{2} + \frac{1}{2} x_k^2}.$$

A similar formula for the pre-image of multiplicative Boolean compound Poisson.

Unimodality.

Proposition.

The only unimodal measures in $\mathcal{L}$ are delta measures and Cauchy distributions.

This provides many examples of measures μ with connected support such that $\mu^{\boxplus t}$ is never unimodal, answering a question of Hasebe and Sakuma.

Relation to Cébron's map I.

In (Cébron 2014), he defined a homomorphism $\mathbf{e}_{\boxplus} : \mathcal{ID}^{\boxplus} \to \mathcal{ID}^{\boxtimes}$ which satisfies

$$W = \mathcal{BP}_{\boxtimes \to \circledast} \circ \mathbf{e}_{\boxplus} \circ \mathcal{BP}_{* \to \boxplus}.$$

He also proved that

$$\mathbf{e}_{\boxplus}(\mu) = \lim_{n \to \infty} \left(W(\mu^{\boxplus \frac{1}{n}}) \right)^{\boxtimes n}.$$

Thus on $\mathcal{ID}^{\boxplus} \cap \mathcal{L}$, $\mathbf{e}_{\boxplus} = W$. He also observed that W (roughly speaking) wraps the Lévy measure of μ. Therefore on $\mathcal{L}$, it does the same with its free, Boolean, monotone Lévy measures.

Relation to Cébron's map II.

Example.

Let ν be the multiplicative free Gaussian measure. Of course $\mathbf{e}_{\boxplus}(\sigma) = \nu$ for the semicircular distribution σ, with canonical pair

$$(0, \delta_0).$$

But $\sigma \notin \mathcal{L}$, and $W(\sigma) \neq \nu$. Instead, $W(\mu) = \nu$ for $\mu \in \mathcal{L}$ with canonical pair

$$\left(\sum_{k \neq 0} \frac{1}{2\pi k (1 + (2\pi k)^2)}, \sum_{k \in \mathbb{Z}} \frac{1}{1 + (2\pi k)^2} \delta_{2\pi k} \right).$$

Corollary.

W intertwines the restrictions of the Bercovici-Pata maps to $\mathcal{L}$ with their multiplicative counterparts.

Obvious properties of W.

- W sends atoms to atoms.
- If $\mathrm{supp}(\mu^{ac}) = \mathbb{R}$, then $\mathrm{supp}((W(\mu))^{ac}) = \mathbb{T}$.
- W sends infinitesimal triangular arrays $\{\mu_{ni}, 1 \leqslant i \leqslant k_n\}_{n \in \mathbb{N}}$ of measures in $\mathcal{P}(\mathbb{R})$ to infinitesimal triangular arrays of measures in $\mathcal{P}(\mathbb{T})$.

Converses.

Theorem.

- For $\mu \in \mathcal{L}$, W maps the atoms of μ bijectively onto the atoms of $W(\mu)$, and preserves the weights.
- If $\operatorname{supp}((W(\mu))^{ac}) = \mathbb{T}$, then $\operatorname{supp}(\mu^{ac}) = \mathbb{R}$.
- If $\{\nu_{ni}, 1 \leqslant i \leqslant k_n\}_{n \in \mathbb{N}}$ is an infinitesimal triangular arrays of measures in $\mathcal{ID}_*^{\circledast}$, then $\nu_{ni} = W(\mu_{ni})$ for some infinitesimal triangular array of measures in $\mathcal{L}$.
- For $\mu \in \mathcal{L}$, F_μ is injective if and only if $\eta_{W(\mu)}$ is.

Corollaries I.

Can re-prove many results, but only for measures in $\mathcal{ID}^{\cup\!\!\!\!\times}_*$.

- Atoms, singular continuous part, components of the absolutely continuous part of $\mu \boxtimes \nu$. Some of these are new.
- Atoms, singular continuous part, components of the absolutely continuous part of $\nu^{\boxtimes}$. These only make sense for $\nu \in \mathcal{ID}^{\cup\!\!\!\!\times}_*$.

Corollary.

Let $\nu \in \mathcal{ID}^{\cup\!\!\!\!\times}_*$ and $t \geqslant 1$. Denote $\nu_t = \nu^{\boxtimes t}$. ζ is an atom of ν_t if and only if for some $\alpha \in \mathbb{R}$, $e^{-it\alpha} = \zeta$ and $e^{-i\alpha}$ is an atom of ν, with $\nu(\{e^{-i\alpha}\}) > 1 - 1/t$, in which case

$$\nu_t(\{\zeta\}) = t\nu(\{e^{-i\alpha}\} - (t-1).$$

Corollaries II.

- Limit theorems for $\circledast, \boxtimes, \times\!\!\!\!\cup, \circlearrowright$ from those for $*, \boxplus, \uplus, \triangleright$.
- First examples of limit theorems for non-identically distributed monotone arrays beyond the finite variance case.

Thank you!