Obstructions to embedding subsets of Schatten classes in L_p spaces

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Joint work with

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Bi-Lipschitz embedding

A metric space (X, d_X) is said to admit a bi-Lipschitz embedding into a metric space (Y, d_Y) if there exist $s \in (0, \infty)$, $D \in [1, \infty)$ and a mapping $f : X \to Y$ such that

$$\forall\, x, y \in X, \qquad s d_X(x, y) \le d_Y(f(x), f(y)) \le D s d_X(x, y).$$

When this happens we say that (X, d_X) embeds into (Y, d_Y) with distortion at most D. We denote by $c_Y(X)$ the infinum over such $D \in [1, \infty]$. When $Y = L_p$ we use the shorter notation $c_{L_p}(X) = c_p(X)$.

We are interested in bounding from below the distortion of embedding certain metric spaces into L_p. I'll concentrate on embedding certain grids in Schatten p-classes into L_p.

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Schatten classes

Given a (finite or infinite, real or complex) matrix A and $1 \leq p < \infty$

$$\|A\|_p = (\mathrm{trace}(A^*A)^{p/2})^{1/2} = (\sum_{i=1}^{\infty} \lambda_i^p)^{1/p}$$

where the λ_i-s are the singular values of A.

$$\|A\|_\infty = \|A : \ell_2 \to \ell_2\|.$$

S_p^n is the space of all $n \times n$ matrices equipped with the norm $\| \cdot \|_p$.

e_{ij} denotes the matrix with 1 in the ij place and zero elsewhere.
This is a good basis in a certain order but, except if $p = 2$, NOT a good unconditional basis.

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Recall that $\{x_i\}_{i=1}^m \subset X$ is a K-unconditional if for all (say real) scalars $\{a_i\}_{i=1}^m$ and signs $\{\varepsilon_i\}_{i=1}^m$,

$$\|\sum a_i x_i\| \le K \|\sum \varepsilon_i a_i x_i\|.$$

Here is a simple way to show that e_{ij} is not a good unconditional basis. For simplicity, $p = 1$.

Claim

$$\mathbb{E}_{\varepsilon_{ij}=\pm 1} \|\sum_{i,j=1}^n \varepsilon_{ij} e_{ij}\|_1 \approx n^{3/2},$$

While

$$\|\sum_{i,j=1}^n e_{ij}\|_1 = n.$$

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Schatten classes

The $\geq$ side in the first equivalence follows easily from duality between S_1^n and S_∞^n and the not-hard fact that

$$\mathbb{E}_{\varepsilon_{ij}=\pm 1}\|\sum_{i,j=1}^{n}\varepsilon_{ij}e_{ij}\|_\infty \lesssim n^{1/2}.$$

Note also that for all $\varepsilon_i, \delta_j = \pm 1$ $\quad \|\sum_{i,j=1}^{n}\varepsilon_i\delta_j e_{ij}\|_1 = n$.
So, the best constant K in the inequality

$$\mathbb{E}_{\varepsilon_{ij}=\pm 1}\|\sum_{i,j=1}^{n}\varepsilon_{ij}x_{ij}\|_1 \leq K\mathbb{E}_{\varepsilon_i,\delta_j=\pm 1}\|\sum_{i,j=1}^{n}\varepsilon_i\delta_j x_{ij}\|_1$$

holding for all $\{x_{ij}\}$ in S_1 is at least of order $n^{1/2}$.
On the other hand, it follows from Khinchine's inequality that for all $\{x_{ij}\}$ in L_1,

$$\mathbb{E}_{\varepsilon_{ij}=\pm 1}\|\sum_{i,j=1}^{n}\varepsilon_{ij}x_{ij}\|_1 \lesssim \mathbb{E}_{\varepsilon_i,\delta_j=\pm 1}\|\sum_{i,j=1}^{n}\varepsilon_i\delta_j x_{ij}\|_1. \quad (\textit{upper property } \alpha)$$

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non-linear embeddings

It follows that the Banach–Mazur distance of S_1^n from a subspace of L_1 (or any other space with "upper property α") is at least of order $n^{1/2}$. It is easy to see that this is the right order.
It follows from general principles (mostly differentiation) that $c_p(S_1^n)$ is equal to their linear counterparts. But these principles no longer apply when dealing with $c_p(A)$ for a discrete set $A \subset S_1^n$
nor for $c_p((S_1^n)^a)$ where for $0 < a < 1$ $(S_1^n)^a$ denotes S_1^n with the metric $d_a(x, y) = \|x - y\|_1^a$.

Our purpose is to find an inequality similar to the upper property α inequality but which will involve only distances between pairs of points and which holds in L_1 but grossly fails in S_1^n.

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Digression: Enflo's type

A metric space (X, d_X) is said to have (Enflo) type $r \in [1, \infty)$ if for every $n \in \mathbb{N}$ and $f : \{-1, 1\}^n \to X$,

$$\mathbb{E}\left[d_X(f(\varepsilon), f(-\varepsilon))^r\right] \lesssim \sum_{j=1}^{n} \mathbb{E}\left[d_X(f(\varepsilon), f(\varepsilon_1, \ldots, \varepsilon_{j-1}, -\varepsilon_j, \varepsilon_{j+1}, \ldots, \varepsilon_n))^r\right], \quad (1)$$

where the expectation is with respect to $\varepsilon \in \{-1, 1\}^n$ chosen uniformly at random. Note that if X is a Banach space and f is the linear function given by $f(\varepsilon) = \sum_{j=1}^{n} \varepsilon_j x_j$ then this is the inequality defining type r:

$$\mathbb{E}\|\sum_{j=1}^{n} \varepsilon_j x_j\|^r \lesssim \sum_{j=1}^{n} \|x_j\|^r$$

For $p \in [1, \infty)$, L_p actually has Enflo type $r = \min\{p, 2\}$. i.e., $X = L_p$ satisfies (1) with $f : \{-1, 1\}^n \to L_p$ allowed to be an arbitrary mapping rather than only a linear mapping.

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This statement was proved by Enflo in 1969 for $p \in [1,2]$ (and by [NS, 2002] for $p \in (2,\infty)$).

Here is an illustration how to use Enflo type to show that for $q < p \le 2$ $c_p(\{-1,1\}^n, \|\cdot\|_q) \gtrsim n^{\frac{1}{q}-\frac{1}{p}}$ ($c_p(\ell_q^n) \le n^{\frac{1}{q}-\frac{1}{p}}$ is trivial).

Let $f : \{-1,1\}^n \to L_p$ be such that

$$\forall\, x, y \in \{-1,1\}^n, \qquad \|x-y\|_q \le \|f(x)-f(y)\|_p \le D\|x-y\|_q$$

Then

$$2^p n^{p/q} \le \mathbb{E}\|f(\varepsilon)-f(-\varepsilon)\|_p^p \lesssim \sum_{j=1}^n \mathbb{E}\|f(\varepsilon)-f(\varepsilon_1,\ldots,\varepsilon_{j-1},-\varepsilon_j,\varepsilon_{j+1},\ldots,\varepsilon_n)\|_p^p \lesssim D^p n 2^p.$$

So $D \gtrsim n^{\frac{1}{q}-\frac{1}{p}}$.

Similarly one shows that for $\alpha > q/p$ $c_p(\{-1,1\}^n, \|\cdot\|_q^\alpha) \to \infty$.

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The definition of non-linear cotype is more problematic. Changing the direction of the inequality in the definition of type is no good if $f(\{-1,1\}^n)$ is a discrete set. A good definition was sought for a long time until the following:

A metric space (X, d_X) is said to have (Mendel-Naor) cotype $s \in [1, \infty)$ if for every $n \in \mathbb{N}$ there is an $m \in \mathbb{N}$ such that for all $f : \mathbb{Z}_{2m}^n \to X$,

$$\sum_{j=1}^{n} \frac{\mathbb{E}\left[d_X(f(x+me_j), f(x))^s\right]}{m^s} \lesssim \mathbb{E}\left[d_X(f(x+\varepsilon), f(x))^s\right],$$

where the expectation is with respect to $(x, \varepsilon) \in \mathbb{Z}_{2m}^n \times \{-1,0,1\}^n$ chosen uniformly at random.

The definition of non-linear cotype is more problematic. Changing the direction of the inequality in the definition of type is no good if $f(\{-1,1\}^n)$ is a discrete set. A good definition was sought for a long time until the following:

A metric space (X, d_X) is said to have (Mendel-Naor) cotype $s \in [1, \infty)$ if for every $n \in \mathbb{N}$ there is an $m \in \mathbb{N}$ such that for all $f : \mathbb{Z}_{2m}^n \to X$,

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back to non-linear version of upper property α

We are looking for a good non-linear version of the *linear* upper α inequality:

$$\mathbb{E}_{\varepsilon_{ij}=\pm 1}\|\sum_{i,j=1}^{n}\varepsilon_{ij}x_{ij}\| \le K\,\mathbb{E}_{\varepsilon_i,\delta_j=\pm 1}\|\sum_{i,j=1}^{n}\varepsilon_i\delta_j x_{ij}\|.$$

We denote by $\alpha(X)$ the best K which works for all x_{ij} -s in the normed space X.

We want to find obstructions to embedding of the grid $M_n[m]$ of all $n\times n$ matrices with values in $[m]=\{-m,-(m-1),\dots,m-1,m\}$ with the S_1 norm (more generally the S_p norm, $1\le p<2$) in a Banach space X with upper property α. In particular L_1 (or L_p).

Something like the following comes to mind: For all $f: M_n[m]\to X$,

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This inequality is problematic and wrong even for $X = \mathbb{R}$ because of the summation over different regions in the right and left sides.

There are (at least) two ways one can try to overcome this: either by wrapping [m] around, i.e. regarding summation mod $2m + 1$. Or by some “smoothing” of the inequality, as will be explained later.

The first method leads to elegant inequalities having to do with expansion properties of a natural graph, but unfortunately we do not see a way to use them to prove our main concern: that $M_n[m]$ with the S_1^n distance does not nicely Lipschitz embed into L_1.

The second methods leads to a solution to our problem (but as we'll see the resulting inequality is not so elegant).

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Binary tensor conductance of $M_n(\mathbb{Z}_m)$

$\mathbb{Z}_m$ denotes $\{0, 1, \dots m-1\}$ with addition mod m.

Theorem

Let $m, n \in \mathbb{N}$, $1 \le p < \infty$, with $n^6 \lesssim_p m$ and let X be a Banach space. Let $f : M_n(\mathbb{Z}_m) \to X$ be any function. Then

$$\mathbb{E}_{x,y\in M_n(\mathbb{Z}_m)} \|f(x)-f(y)\|^p \lesssim_p \alpha(X) m^p \, \mathbb{E}_{\substack{x\in M_n(\mathbb{Z}_m)\\ \varepsilon,\delta\in\{0,1\}^n}} \|f(x+\varepsilon\otimes\delta)-f(x)\|^p.$$

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metric upper α inequality

Theorem

For every normed space X and all n, k and m satisfying $n^6\alpha(X) \le k \le C\min\{m^2/(n^6\alpha(X)), m/n^2\}$, there is an $M > m$ with $M/m \to 1$ as $n \to \infty$ such that for all $f : \mathbb{Z}^{n^2} \to X$,

$$\mathbb{E}_{\substack{x\in M_n[m] \\ \varepsilon\in M_n(\{-1,1\})}} \|f(x+8k\varepsilon) - f(x)\|^p \lesssim_p k^p\alpha^p(X)\,\mathbb{E}_{\substack{x\in M_n[M] \\ \varepsilon,\delta\in\{-1,1\}^n}} \|f(x+\varepsilon\otimes\delta) - f(x)\|^p.$$

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Conversely, Assume that a Banach space X satisfy the inequality,

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Fixing $\{y_{ij}\}\subset X$ and applying the inequality to $f(x)=\sum_{ij}x_{ij}y_{ij}$, we get

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Claim

For any n and M large enough with respect to n, the distortion of embedding $M_n(M)$ with the S_1 distance into a Banach space X is, at least of order $n^{1/2}/\alpha(X)$.

Proof: If $f : M_n[M] \to X$ is such that

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