Free Multiplicative Brownian Motion, and Brown Measure

Extended Probabilistic Operator Algebras Seminar
UC Berkeley

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UC San Diego

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Giving Credit where Credit is Due

Based partly on joint work with Bruce Driver and Brian Hall, and highlighting the work of Philippe Biane.

- Biane, P.: *Free Brownian motion, free stochastic calculus and random matrices*. Fields Inst. Commun. vol. 12, Amer. Math. Soc., PRovidence, RI, 1-19 (1997)
- Biane, P.: *Segal-Bargmann transform, functional calculus on matrix spaces and the theory of semi-circular and circular systems*. J. Funct. Anal. 144, 1, 232-286 (1997)
- Driver; Hall; K: *The large-N limit of the Segal–Bargmann transform on $\mathbb{U}_N$*. J. Funct. Anal. 265, 2585-2644 (2013)
- K: *The Large-N Limits of Brownian Motions on $\mathbb{GL}_N$*. Int. Math. Res. Not. IMRN, no. 13, 4012-4057 (2016)
- K: *Heat kernel empirical laws on $\mathbb{U}_N$ and $\mathbb{GL}_N$*. J. Theoret. Probab. 30, no. 2, 397-451 (2017)

Brownian Motion on $\mathrm{U}(N)$, $\mathrm{GL}(N, \mathbb{C})$, and the Large-$N$ Limit

Brownian Motion on Lie Groups

On any Riemannian manifold M, there's a Laplace operator Δ_M. And where there's a Laplacian, there's a Brownian motion: the Markov process $(B_t^x)_{t\geq 0}$ on M with generator $\frac{1}{2}\Delta_M$, started at $B_0^x = x \in M$.

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Let Γ be a (matrix) Lie group. Any inner product on $\mathrm{Lie}(\Gamma) = T_I\Gamma$ gives rise to a unique left-invariant Riemannian metric, and corresponding Laplacian Δ_Γ. On Γ we canonically start the Brownian motion $(B_t)_{t\geq 0}$ at $I \in \Gamma$.

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There is a beautiful relationship between the Brownian motion W_t on the Lie algebra $\mathrm{Lie}(\Gamma)$ and the Brownian motion B_t: the *rolling map*

$$dB_t = B_t \circ dW_t, \qquad \text{i.e.} \qquad B_t = I + \int_0^t B_t \circ dW_t.$$

Here $\circ$ denotes the Stratonovich stochastic integral. This can always be converted into an Itô integral; but the answer depends on the structure of the group Γ (and the chosen inner product).

Brownian Motion on $\mathrm{U}(N)$ and $\mathrm{GL}(N, \mathbb{C})$

Fix the *reverse normalized* Hilbert–Schmidt inner product on $\mathbb{M}_N(\mathbb{C})$ for all matrix Lie algebras:

$$\langle A, B\rangle = N\mathrm{Tr}(B^* A).$$

Let $X_t = X_t^N$ and $Y_t = Y_t^N$ be independent Hermitian Brownian motions of variance t/N.

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The Brownian motion on $\mathrm{Lie}(\mathrm{U}(N))$ is iX_t; the Brownian motion U_t on $\mathrm{U}(N)$ satisfies

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The Brownian motion on $\mathrm{Lie}(\mathrm{GL}(N,\mathbb{C})) = \mathbb{M}_N(\mathbb{C})$ is $Z_t = 2^{-1/2}i(X_t + iY_t)$; the Brownian motion G_t on $\mathrm{GL}(N,\mathbb{C})$ satisfies

$$\boxed{dG_t = G_t\,dZ_t.}$$

Free Additive Brownian Motion

If $X_t = X_t^N$ is a Hermitian Brownian motion process, then at each time $t > 0$ it is a GUE_N with entries of variance t/N. Wigner's law then shows that the empirical spectral distribution of X_t^N converges to the semicircle law $\varsigma_t = \frac{1}{2\pi t}\sqrt{(4t - x^2)_+}\, dx$.

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A process $(x_t)_{t\geq 0}$ (in a W^*-probability space with trace τ) is a **free additive Brownian motion** if its increments are freely independent — $x_t - x_s$ is free from $\{x_r : r \leq s\}$ — and $x_t - x_s$ has the semicircular distribution ς_{t-s}, for all $t > s$.

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In 1991, Voiculescu showed that the Hermitian Brownian motion $(X_t^N)_{t\geq 0}$ converges to $(x_t)_{t\geq 0}$ in finite-dimensional non-commutative distributions:

$$\frac{1}{N}\mathrm{Tr}(P(X_{t_1},\ldots,X_{t_n})) \to \tau(P(x_{t_1},\ldots,x_{t_n})) \qquad \forall P.$$

Free Unitary and Free Multiplicative Brownian Motion

There is now a well-developed theory of free stochastic differential equations. Initially constructed in the free Fock space setting (by Kümmerer and Speicher in the early 1990s), it was used by Biane in 1997 to define "free versions" of U_t and G_t.

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Let x_t, y_t be freely independent free additive Brownian motions, and $z_t = 2^{-1/2}i(x_t + iy_t)$. The **free unitary Brownian motion** is the process started at $u_0 = 1$ defined by

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It is natural to expect that these processes should be the large-N limits of the $\mathrm{U}(N)$ and $\mathrm{GL}(N, \mathbb{C})$ Brownian motions.

Free Unitary Brownian Motion

Theorem. [Biane, 1997] For all non-commutative (Laurent) polynomials P in n variables and times $t_1, \ldots, t_n \geq 0$,

$$\frac{1}{N}\mathrm{Tr}(P(U^N_{t_1}, \ldots, U^N_{t_n})) \to \tau(P(u_{t_1}, \ldots, u_{t_n})) \text{ a.s.}$$

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Biane also computed the moments of u_t, and its spectral measure ν_t: it has a density (smooth on the interior of its support), supported on a compact arc for $t < 4$, and fully supported on $\mathbb{U}$ for $t \ge 4$.

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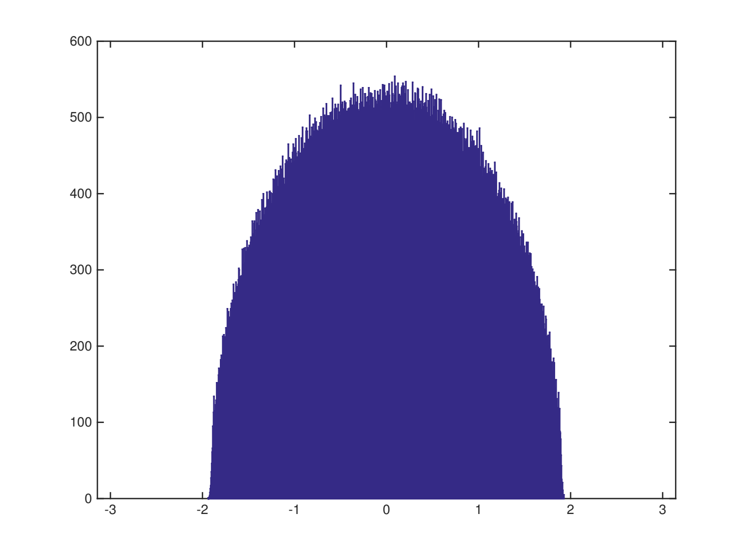

Analytic Transforms Related to u_t

Biane's approach to understanding the measure ν_t was through its moment-generating function

$$\psi_t(z) = \int_{\mathbb{U}} \frac{uz}{1-uz}\,\nu_t(du) = \sum_{n\geq 1} m_n(\nu_t)\,z^n$$

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(the second $=$ holds for $|z|<1$; the integral converges for $1/z \notin \operatorname{supp}\nu_t$). Then define

$$\chi_t(z) = \frac{\psi_t(z)}{1+\psi_t(z)}.$$

The function χ_t is injective on $\mathbb{D}$, and has a one-sided inverse f_t: $f_t(\chi_t(z)) = z$ for $z\in\mathbb{D}$ (but $\chi_t\circ f_t$ is only the identity on a certain region in $\mathbb{C}$; more on this later).

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Using the SDE for u_t and some clever complex analysis, Biane showed that

$$f_t(z) = z e^{\frac{t}{2}\frac{1+z}{1-z}}.$$

The Large-N Limit of G_t^N

In 1997 Biane conjectured a similar large-N limit should hold for the Brownian motion on $\mathrm{GL}(N,\mathbb{C})$, but the ideas of his U_t^N proof (spectral theorem, representation theory of $\mathrm{U}(N)$) did not translate well to the a.s. non-normal process G_t^N.

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Theorem. [K, 2014 (2016)] For all non-commutative Laurent polynomials P in $2n$ variables, and times $t_1,\ldots,t_n \geq 0$,

$$\frac{1}{N}\mathrm{Tr}\left(P(G_{t_1}^N,(G_{t_1}^N)^*,\ldots,G_{t_n}^N,(G_{t_n}^N)^*)\right) \to \tau\left(P(g_{t_1},g_{t_1}^*,\ldots,g_{t_n},g_{t_n}^*)\right) \text{ a.s.}$$

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This is convergence of the (multi-time) $*$-distribution, of a *non-normal* matrix process. What about the eigenvalues?

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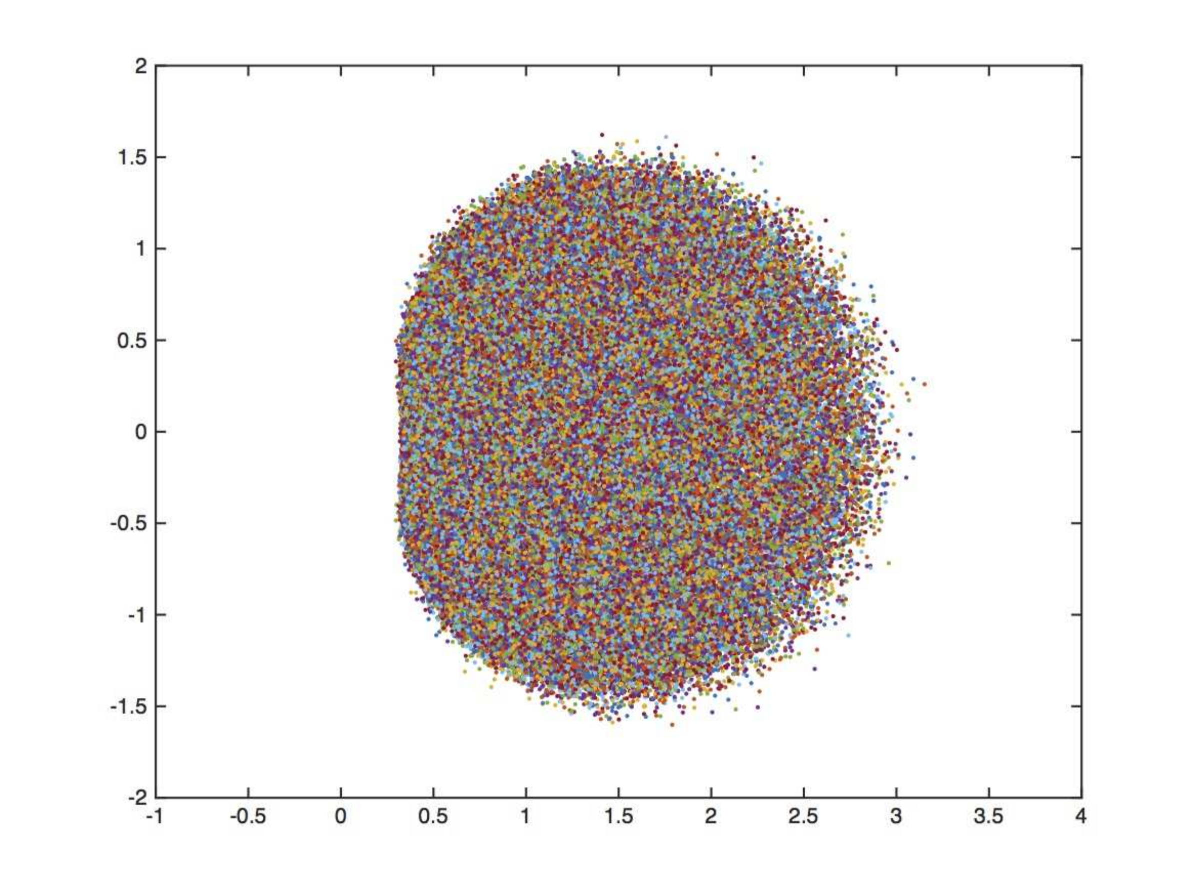

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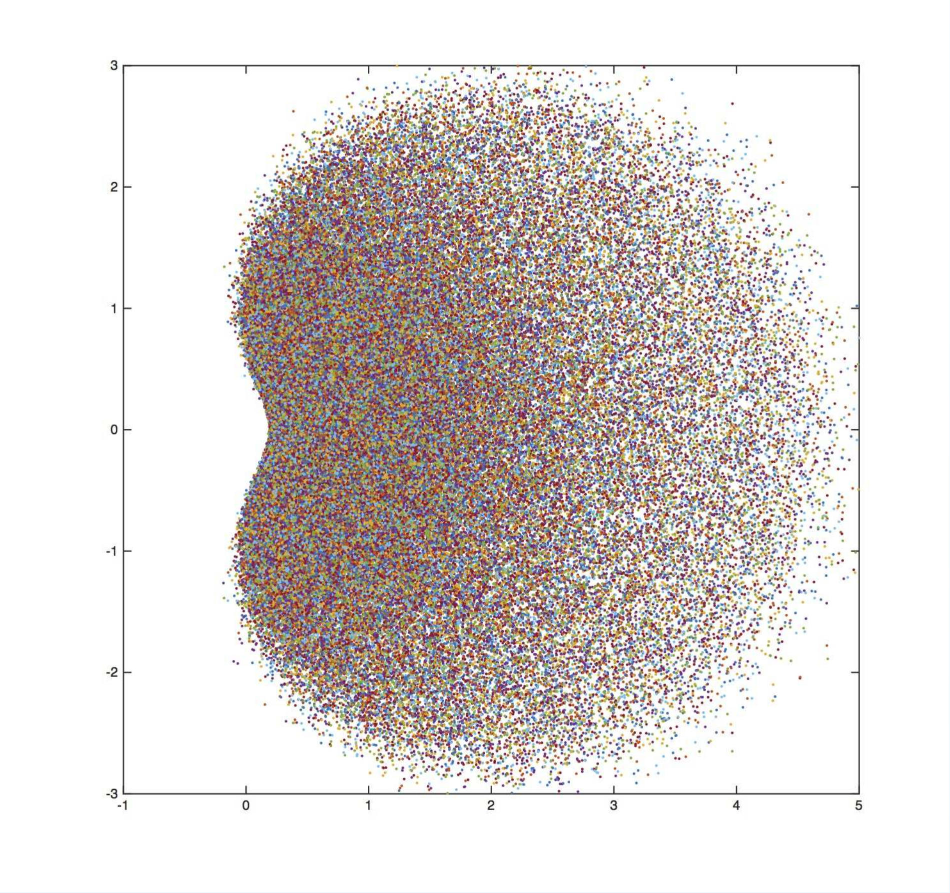

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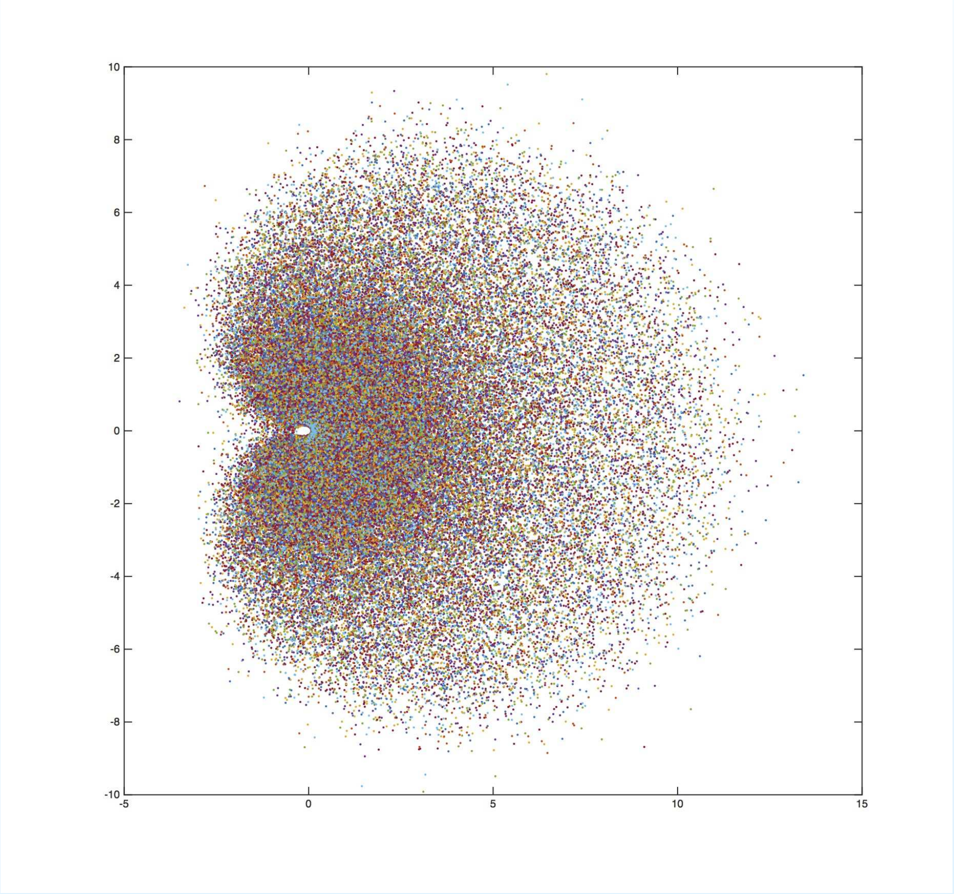

$t = 4$

Brown's Spectral Measure

Brown's Spectral Measure in Tracial von Neumann Algebras

If $(\mathcal{A}, \tau)$ is a W^*-probability space, then any normal operator $a \in \mathcal{A}$ has a spectral measure $\mu_a = \tau \circ E^a$. If A is a normal matrix, μ_A is its ESD. It is characterized (nicely) by the $*$-distribution of a:

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If a is not normal, there is no such measure. But there is a substitute: Brown's spectral measure. Let $L(a)$ denote the (log) Kadison–Fuglede determinant:

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(the last $=$ holds if $a^{-1} \in \mathcal{A}$). Then $\lambda \mapsto L(a - \lambda)$ is subharmonic on $\mathbb{C}$, and

$$\mu_a = \frac{1}{2\pi} \nabla_\lambda^2 L(a - \lambda)$$

is a probability measure on $\mathbb{C}$. If A is *any* matrix, μ_A is its ESD.

Properties of Brown Measure

The Brown measure has some nice properties analogous to the spectral measure, but not all:

- $\tau(a^k) = \int_{\mathbb{C}} z^k \, \mu_a(dzd\bar{z})$ and $\tau(a^{*k}) = \int_{\mathbb{C}} \bar{z}^k \, \mu_a(dzd\bar{z})$ but you *cannot max and match*.

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Convergence of the Brown Measure

Let $\{a, a_n\}_{n\in\mathbb{N}}$ be a uniformly bounded set of operators in some W^*-probability spaces, with $a_n \to a$ in $*$-distribution. We would hope that $\mu_{a_n} \to \mu_a$. Without some very fine information about the spectral measure of $|a_n - \lambda|$ near the edge of $\mathrm{Spec}(a_n)$, the best that can be said in general is the following.

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Proposition. Suppose that $\mu_{a_n} \to \mu$ weakly for some probability measure μ on $\mathbb{C}$. Then

$$\int_{\mathbb{C}} \log|z-\lambda|\,\mu(dzd\bar{z}) \le \int_{\mathbb{C}} \log|z-\lambda|\,\mu_a(dzd\bar{z})$$

for all $\lambda \in \mathbb{C}$; and equality holds for sufficiently large λ.

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Corollary. Let V_a be the unbounded connected component of $\mathbb{C}\setminus \mathrm{supp}\,\mu_a$. Then $\mathrm{supp}\,\mu \subseteq \mathbb{C}\setminus V_a$. (In particular, if $\mathrm{supp}\,\mu_a$ is simply-connected, then $\mathrm{supp}\,\mu \subseteq \mathrm{supp}\,\mu_a$.)

Brown Measure via Regularization

The function $L(a-\lambda) = \int_{\mathbb{R}} \log t \, \mu_{|a|}(dt)$ is essentially impossible to compute with. But we can use regularity properties of the spectral resolution to approach it in a different way. Define

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The function $\lambda \mapsto L^\epsilon(a-\lambda)$ is $C^\infty(\mathbb{C})$, and is subharmonic. Define

$$h_a^\epsilon(\lambda) = \frac{1}{2\pi}\nabla_\lambda^2 L_\epsilon(a-\lambda).$$

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$$\mu_a(d\lambda) = \lim_{\epsilon\downarrow 0} h_a^\epsilon(\lambda)\, d\lambda.$$

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It is not difficult to explicitly calculate the density h_a^ϵ for fixed $\epsilon > 0$.

The Density h_a^ϵ and the Spectrum of a

Lemma. Let $\lambda \in \mathbb{C}$, and denote $a_\lambda = a - \lambda$. Then

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From here it is easy to see why supp $\mu_a \subseteq \mathrm{Spec}(a)$. If $\lambda \in \mathrm{Res}(a)$ so that $a_\lambda^{-1} \in \mathcal{A}$, we quickly estimate

$$\begin{aligned}&\big|\tau\big((a_\lambda^* a_\lambda + \epsilon)^{-1}(a_\lambda a_\lambda^* + \epsilon)^{-1}\big)\big| \\ &\le \big\|(a_\lambda^* a_\lambda + \epsilon)^{-1}(a_\lambda a_\lambda^* + \epsilon)^{-1}\big\|\end{aligned}$$

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This is locally uniformly bounded in λ; so taking $\epsilon \downarrow 0$, the factor of ϵ in $h_a^\epsilon(\lambda)$ kills the term; we find $\mu_a = 0$ in a neighborhood of λ.

Invertibility in $L^p(\mathcal{A}, \tau)$

Recall that $L^p(\mathcal{A}, \tau)$ is the closure of $\mathcal{A}$ in the norm

$$\|a\|_p^p = \tau(|a|^p) = \tau\Big((a^*a)^{p/2}\Big).$$

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From Hölder's inequality, we have the inclusions

$$\mathrm{Spec}_{p,\tau}(a) \subseteq \mathrm{Spec}_{q,\tau}(a) \subseteq \mathrm{Spec}(a)$$

for $1 \leq p \leq q < \infty$. Without including the closure in the definition, these inclusions can be strict; with the closure, my (wild) conjecture is that $\mathrm{Spec}_{1,\tau}(a) = \mathrm{Spec}(a)$ for all a.

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If we naïvely set $\epsilon = 0$ on the right-hand-side, we get (heuristically)

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Note, this is *not* equal to $\|a_\lambda^{-1}\|_4^4$ when a_λ is not normal.

The $L^2_{2,\tau}$ Spectrum

Proposition. Let $a \in \mathcal{A}$, and suppose a^2 is invertible in $L^2(\mathcal{A}, \tau)$. Then for all $\epsilon > 0$,

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Another wild conjecture: this is actually equality. (That depends on showing that, if a^2 is *not* invertible in $L^2(\mathcal{A},\tau)$, the above quantity blows up at rate $\Omega(1/\epsilon)$. This appears to be what happens in the case that a is normal, which would imply $\mathrm{Spec}^2_{2,\tau}(a) = \mathrm{Spec}_{4,\tau}(a)$ $= \mathrm{Spec}(a)$ in that case.)

The Segal–Bargmann Transform

The Unitary Segal–Bargmann Transform

The **Segal–Bargmann (Hall) Transform** is a map from functions on $\mathrm{U}(N)$ to holomorphic functions on $\mathrm{GL}(N, \mathbb{C})$. It is defined by the analytic continuation of the action of the heat operator:

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Writing out what this integral formula means in probabilistic terms, here is a nice way to express it: let F already be a holomorphic function on $\mathrm{GL}(N),\mathbb{C})$, and let $f = F|_{\mathrm{U}(N)}$. Let U_t and G_t be independent Brownian motions on $\mathrm{U}(N)$ and $\mathrm{GL}(N,\mathbb{C})$. Then

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This extends beyond f that already possess an analytic continuation; it defines an *isometric isomorphism*

$$\mathbf{B}_t^N \colon L^2(\mathrm{U}(N), U_t) \to \mathcal{H}L^2(\mathrm{GL}(N,\mathbb{C}), G_t).$$

The Free Unitary Segal–Bargmann Transform

In 1997, Biane introduced a free version of the Unitary SBT, which can be described in similar terms: acting on, say, polynomials f in a single variable, $\mathscr{G}_t f$ is defined by

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He conjectured that $\mathscr{G}_t$ is the large-N limit of $\mathbf{B}_t^N$ in an appropriate sense; this was proven by Driver, Hall, and me in 2013. (It was for this work that we invented trace polynomial concentration.)

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Biane proved directly (and it follows from the large-N limit) that $\mathscr{G}_t$ extends to an isometric isomorphism

$$\mathscr{G}_t \colon L^2(\mathbb{U}, \nu_t) \to \mathscr{A}_t$$

where $\mathscr{A}_t$ is a certain reproducing-kernel Hilbert space of holomorphic functions. The norm on $\mathscr{A}_t$ is given by

$$\|F\|^2_{\mathscr{A}_t} = \tau(|F(g_t)|^2) = \tau(F(g_t)^* F(g_t)) = \|F(g_t)\|_2^2.$$

The Range of the Free Segal–Bargmann Transform

The functions $F \in \mathscr{A}_t$ are not all entire functions. They are holomorphic on a bounded region Σ_t

$$\Sigma_t = \mathbb{C} \setminus \overline{\chi_t(\mathbb{C} \setminus \text{supp}\, \nu_t)}$$

where (recall) χ_t is the (right-)inverse of $f_t(z) = ze^{\frac{t}{2}\frac{1+z}{1-z}}$.

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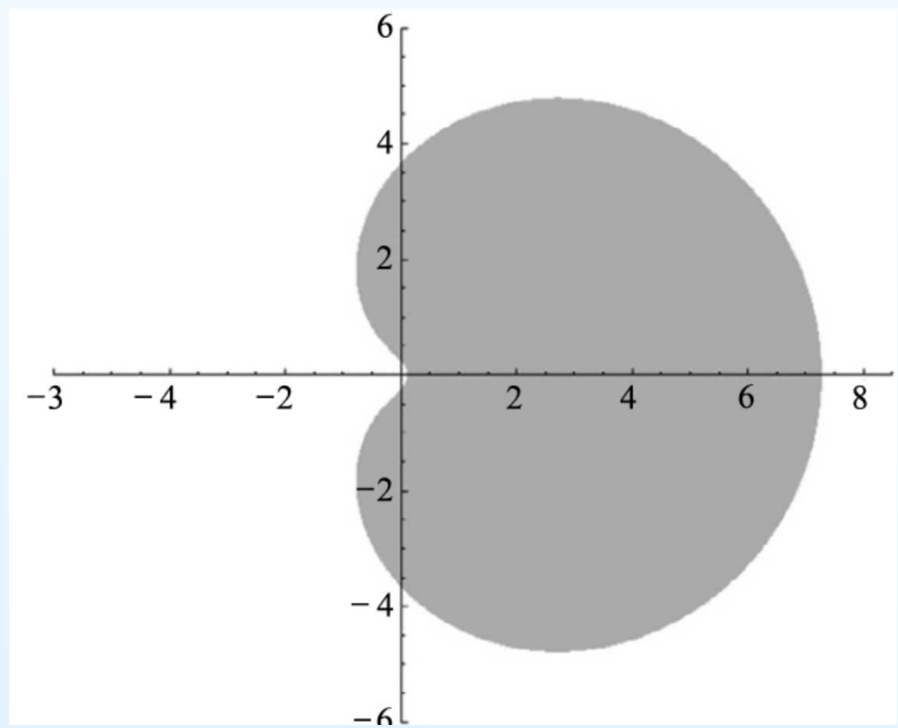

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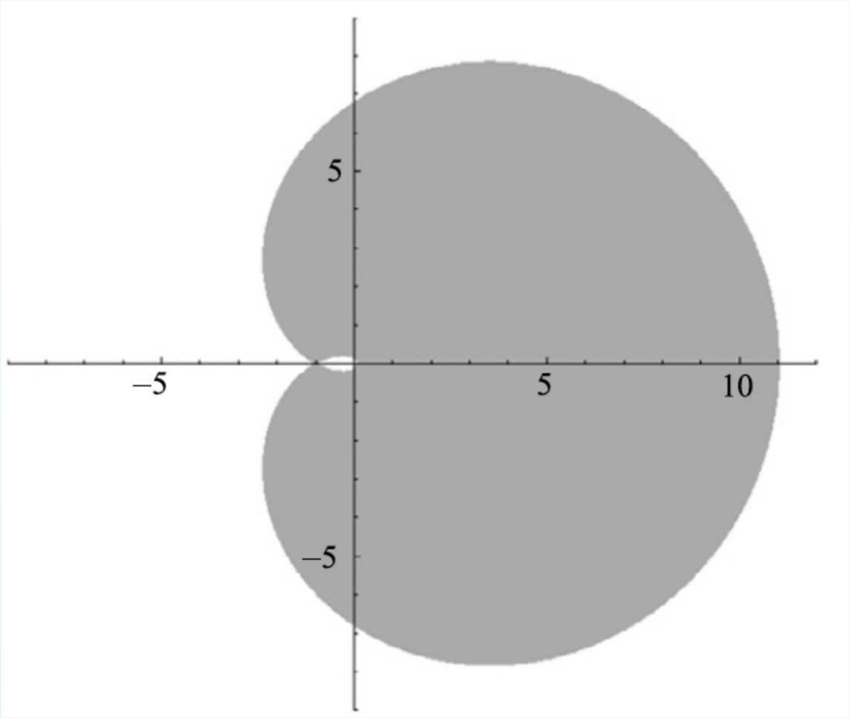

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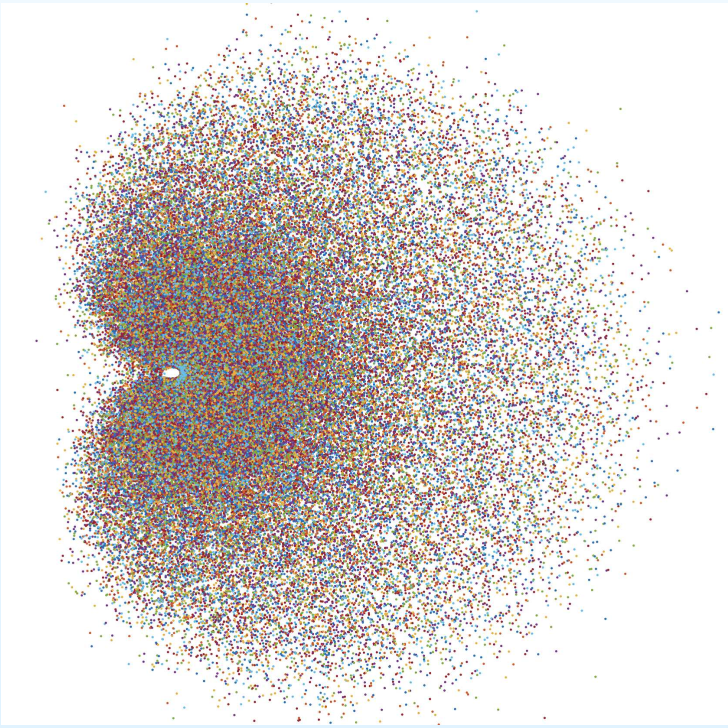

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The Support of The Brown Measure of g_t

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Proof. We show that $\mathrm{Spec}^2_{2,\tau}(g_t) = \overline{\Sigma_t}$. Equivalently, from the definition of Σ_t, we show that $\mathrm{Res}^2_{2,\tau}(g_t) = \chi_t(\mathbb{C} \setminus \text{supp}\, \nu_t)$.

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By definition, $\lambda \in \mathrm{Res}^2_{2,\tau}(g_t)$ iff $(g_t - \lambda)^2$ is invertible in $L^2(\tau)$, i.e.

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Recall that $\mathscr{G}_t$ is an isometry from $L^2(\mathbb{U}, \nu_t)$ onto $\mathscr{A}_t$. Can we find a function α_t^λ on $\mathbb{U}$ with $\mathscr{G}_t(\alpha_t^\lambda)(z) = (z-\lambda)^{-2}$?

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Using PDE techniques, we can compute that

$$\mathscr{G}_t^{-1}((z-\lambda)^{-1}) = \frac{1}{\lambda}\frac{f_t(\lambda)}{f_t(\lambda) - u}.$$

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$$\mathscr{G}_t : \frac{1}{\lambda}\frac{f_t(\lambda)}{f_t(\lambda) - u} \mapsto \frac{1}{z-\lambda}.$$

Since $\frac{1}{(z-\lambda)^2} = \frac{d}{d\lambda}\frac{1}{z-\lambda}$, using regularity properties of $\mathscr{G}_t$ we have

$$\alpha_t^{\lambda}(u) = \frac{d}{d\lambda}\left(\frac{1}{\lambda}\frac{f_t(\lambda)}{f_t(\lambda) - u}\right).$$

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The question is: for which λ is $\alpha_t^\lambda \in L^2(\mathbb{U}, \nu_t)$? I.e.

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The answer is: precisely when $f_t(\lambda) \notin \operatorname{supp}\nu_t$. I.e.

$$\operatorname{Res}^2_{2,\tau}(g_t) = f_t^{-1}(\mathbb{C}\setminus \operatorname{supp}\nu_t) = \chi_t(\mathbb{C}\setminus\operatorname{supp}\nu_t).$$

□

Remaining Questions

- Explore relations between the $L^p(\tau)$-spectra, in general. They are probably all equal to the spectrum for g_t.

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- Prove that the ESD of G_t^N actually converges to μ_{g_t}. (What we can now say definitively is that the limit ESD is supported in Σ_t for $t < 4$; for $t \geq 4$, we need more arguments to rule out eigenvalues inside the inner ring.)

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- There is a three parameter family of invariant diffusions on $\mathrm{GL}(N, \mathbb{C})$ that includes U_t^N and G_t^N, all of which have large-N limits described by free SDEs. How much of all this extends to the whole family?

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I'll let you know what more I know next time we meet. ○