Markovianity and the Thompson Group F

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This talk is on the following connection:

Representations of the Thompson group

$$F = \langle g_0, g_1, \ldots \mid g_k g_l = g_{l+1} g_k, 0 \le k < l < \infty \rangle_{\text{group}}$$

$\updownarrow$

Bilateral noncommutative stationary Markov processes

The Thompson group F was introduced by Richard Thompson in 1965 as a certain subgroup of piece-wise linear homeomorphisms on the interval $[0,1]$.
The generators of F satisfy the relations

$$g_k g_l = g_{l+1} g_k, 0 \leq k < l < \infty.$$

For instance,

$$g_1 g_4 = g_5 g_1.$$

We are interested in certain probabilistic aspects of F; in particular its surprising connection to Markovianity.

Our setting of a **noncommutative probability space (NCPS)** will consist of a pair $(\mathcal{M}, \psi)$, where $\mathcal{M}$ is a von Neumann algebra and ψ is a faithful normal state on $\mathcal{M}$.

Classical Probability Space: Let (Ω, Σ, μ) be a standard probability space. Then $\mathcal{L} := L^\infty(\Omega, \Sigma, \mu)$ is a commutative von Neumann algebra, and

$$\mathrm{tr}_\mu(f) := \int_\Omega f \, du$$

defines a faithful normal tracial state on $\mathcal{L}$.
The pair $(\mathcal{L}, \mathrm{tr}_\mu)$ is a noncommutative probability space.

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Definition

An **automorphism** α of a noncommutative probability space $(\mathcal{M}, \psi)$ is a $*$-automorphism on $\mathcal{M}$ satisfying the stationarity property

$$\psi \circ \alpha = \psi$$

The group of automorphisms of $(\mathcal{M}, \psi)$ will be denoted by $\mathrm{Aut}(\mathcal{M}, \psi)$.

Definition

A bilateral noncommutative **stationary process** $(\mathcal{M}, \psi, \alpha, \mathcal{A}_0)$ consists of a noncommutative probability space $(\mathcal{M}, \psi)$, a ψ-conditioned subalgebra $\mathcal{A}_0 \subset \mathcal{M}$, and an automorphism $\alpha \in \mathrm{Aut}(\mathcal{M}, \psi)$.

The term ψ-conditioned refers to the fact that the unique normal conditional expectation E_0 from $\mathcal{M}$ onto $\mathcal{A}_0$ exists with $\psi \circ E_0 = \psi$.

Stationary Sequence: A stationary process generates a sequence of injective $*$-homomorphisms $\iota_n : \mathcal{A}_0 \to \mathcal{M}$ given by

$$\iota_n := \alpha^n \iota_0, \quad n \in \mathbb{N}_0$$

where ι_0 is the canonical inclusion of $\mathcal{A}_0$ into $\mathcal{M}$.

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Definition

The bilateral noncommutative stationary process $(\mathcal{M}, \psi, \alpha, \mathcal{A}_0)$ is called a (bilateral noncommutative) **stationary Markov process** if for

$$\mathcal{A}_{(-\infty,0]} := \bigvee_{i \in \mathbb{N}_0} \alpha^{-i}(\mathcal{A}_0),$$

$$\mathcal{A}_{[0,\infty)} := \bigvee_{i \in \mathbb{N}_0} \alpha^{i}(\mathcal{A}_0),$$

$$\mathcal{A}_{[0,0]} := \mathcal{A}_0,$$

and E_I denoting the conditional expectation onto $\mathcal{A}_I$, we have

$$E_{(-\infty,0]} \circ E_{[0,\infty)} = E_{[0,0]}.$$

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and E_I denoting the conditional expectation onto $\mathcal{A}_I$, we have

$$\underbrace{E_{(-\infty,0]}}_{\text{past}} \circ \underbrace{E_{[0,\infty)}}_{\text{future}} = \underbrace{E_{[0,0]}}_{\text{present}}$$

Definition

Let $(\mathcal{M}, \psi, \alpha, \mathcal{A}_0)$ be a stationary Markov process and ι_0 be the inclusion map of $\mathcal{A}_0$ into $\mathcal{M}$. Let $T := \iota_0^* \alpha \iota_0$. Then T is called the transition operator associated to the Markov process.

Proposition (Kümmerer 85, 86)

T satisfies the following properties.

1. $\iota_0^* \alpha^n \iota_0 = T^n$ *for all* $n \in \mathbb{N}_0$.
2. *Let* $\iota_n := \alpha^n \iota_0$, $k_1 < k_2 < \cdots < k_n \in \mathbb{N}_0$ *and* $a_1, \ldots, a_n \in \mathcal{A}_0$, $n \in \mathbb{N}$. *Then*

$$\psi\left(\iota_{k_1}(a_1) \cdots \iota_{k_n}(a_n)\right) = \psi\left(a_1 T^{k_2 - k_1}(a_2 T^{k_3 - k_2}(a_3 \cdots T^{k_n - k_{n-1}}(a_n) \cdots))\right).$$

Here $(\mathcal{M}, \psi, \alpha, \iota_0)$ is called a *Markov dilation* of T and $\{\iota_n\}_{n \in \mathbb{N}_0}$ is called a *stationary Markov sequence*.

We will now prove the following connection:

Representations of the Thompson group

$$F = \langle g_0, g_1, \ldots \mid g_k g_l = g_{l+1} g_k, 0 \leq k < l < \infty \rangle_{\text{group}}$$

$$\downarrow$$

Bilateral noncommutative stationary Markov processes

Markov processes from representations of F

Suppose a noncommutative probability space $(\mathcal{M}, \psi)$ is equipped with a representation $\rho : F \to \mathrm{Aut}(\mathcal{M}, \psi)$.

Let $\alpha_0 := \rho(g_0), \alpha_1 := \rho(g_1), \dots, \alpha_n := \rho(g_n), \dots \in \mathrm{Aut}(\mathcal{M}, \psi)$, with fixed point algebras

$$\mathcal{M}^{\alpha_n} := \{x \in \mathcal{M} \mid \alpha_n(x) = x\} \qquad (n \in \mathbb{N}_0).$$

The intersections of fixed point algebras

$$\mathcal{M}_n := \bigcap_{k \geq n+1} \mathcal{M}^{\alpha_k}$$

give the tower of von Neumann subalgebras

$$\mathcal{M}^{\rho(F)} \subset \mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \dots \subset \mathcal{M}_\infty := \bigvee_{n \in \mathbb{N}_0} \mathcal{M}_n \subset \mathcal{M}.$$

Theorem (Köstler & K. arXiv:2204.03595)

Suppose $\rho : F \to \mathrm{Aut}(\mathcal{M}, \psi)$ *is a representation with* $\alpha_m := \rho(g_m)$, *for* $m \in \mathbb{N}_0$. *Let* $\mathcal{M}_0 := \bigcap_{k \geq 1} \mathcal{M}^{\alpha_k}$. *Then* $\big(\mathcal{M}, \psi, \alpha_0, \mathcal{M}_0\big)$ *is a bilateral stationary Markov process.*

In fact, we get a family of stationary noncommutative Markov processes from a representation of F.

Theorem

Suppose $\rho : F \to \mathrm{Aut}(\mathcal{M}, \psi)$ *is a representation with* $\alpha_m = \rho(g_m)$, *for* $m \in \mathbb{N}_0$. *Let* $\mathcal{M}_n := \bigcap_{k \geq n+1} \mathcal{M}^{\alpha_k}$. *Then the quadruple* $\big(\mathcal{M}, \psi, \alpha_m, \mathcal{M}_n\big)$ *is a bilateral stationary Markov process for any* $0 \leq m \leq n < \infty$.

An Example of a Representation of F

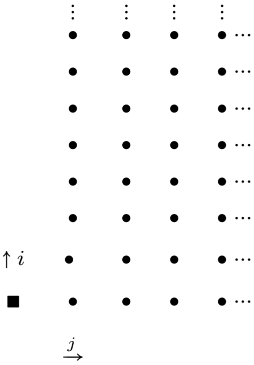
Let's now see an example of a representation ρ of F in the group of automorphisms of a noncommutative probability space, $\mathrm{Aut}(\mathcal{M}, \psi)$.
Let two NCPSs $(\mathcal{A}, \varphi)$ and $(\mathcal{C}, \chi)$ be given.

We build from them the larger NCPS $(\mathcal{M}, \psi)$ given by

$$(\mathcal{M}, \psi) := \Big(\mathcal{A} \otimes \bigotimes_{(i,j)\in\mathbb{N}_0^2} \mathcal{C}_{ij}, \varphi \otimes \bigotimes_{(i,j)\in\mathbb{N}_0^2} \chi_{ij}\Big),$$

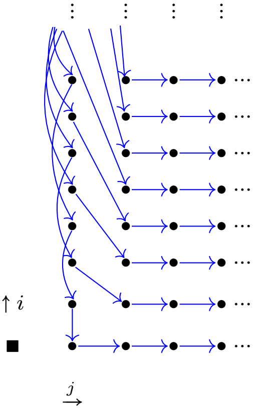
with $\mathcal{C}_{ij} = \mathcal{C}$ and $\chi_{ij} = \chi$ for all $(i,j) \in \mathbb{N}_0^2$.

First let us visualize the von Neumann algebra $\mathcal{M}$ via the set $\{\blacksquare\} \cup \mathbb{N}_0^2$ represented as follows:

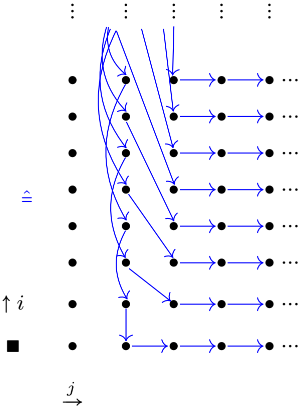


Consider the 'shifts' β_0 and β_1 represented visually on the set $\{\blacksquare\} \cup \mathbb{N}_0^2$ as follows:

$\beta_0 \quad \hat{=}$



and $\beta_1 \quad \hat{=}$



The shifts β_0 and β_1 extend to automorphisms of $(\mathcal{M}, \psi) := \left(\mathcal{A} \otimes \mathcal{C}^{\otimes_{\mathbb{N}_0^2}}, \varphi \otimes \chi^{\otimes_{\mathbb{N}_0^2}}\right)$.
We will define $\rho(g_0) := \beta_0$ and $\rho(g_1) := \beta_1$. Recall that the generators of F satisfy

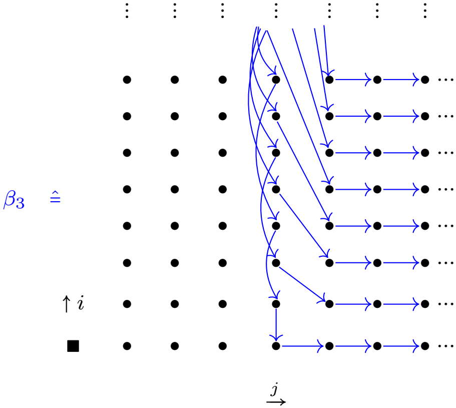
$$g_k g_l = g_{l+1} g_k, \quad k < l.$$

In particular,

$$g_n = g_0^{n-1} g_1 g_0^{-(n-1)}, \quad n > 1.$$

So we define $\beta_n := \beta_0^{n-1} \beta_1 \beta_0^{-(n-1)}$ for $n > 1$.

Here is β_3 visualized, for example:



Fact: The family of automorphisms $\{\beta_n\}$ then satisfy *all* the relations of the Thompson group, so that $\rho(g_n) := \beta_n$ defines a representation ρ of F in $\mathrm{Aut}(\mathcal{M}, \psi)$.

A variation: "Coupling to a Shift"

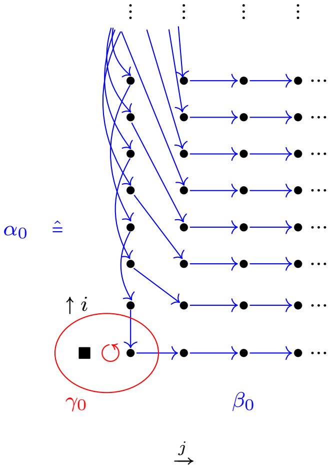
We next obtain another representation of F by "perturbing" the shifts β_n. Given an automorphism $\gamma \in \mathrm{Aut}(\mathcal{A} \otimes \mathcal{C}, \varphi \otimes \chi)$, let $\gamma_0 \in \mathrm{Aut}(\mathcal{M}, \psi)$ denote its natural extension such that

$$\gamma_0 \left(a \otimes \Big(\bigotimes_{(i,j) \in \mathbb{N}_0^2} x_{i,j} \Big) \right) = \gamma(a \otimes x_{00}) \otimes \Big(\bigotimes_{(i,j) \in \mathbb{N}_0^2 \setminus \{(0,0)\}} x_{i,j} \Big).$$

Let $\alpha_0 := \gamma_0 \circ \beta_0$, and $\alpha_n := \beta_n \quad (n \geq 1)$. The perturbation of β_0 to give α_0 as $\alpha_0 = \gamma_0 \circ \beta_0$ is known in the literature as a "coupling to a shift" (Kümmerer 1985).

We have $\alpha_0 := \gamma_0 \circ \beta_0$, and $\alpha_n := \beta_n \quad (n \geq 1)$.

The perturbed shift α_0 on the set $\{\blacksquare\} \cup \mathbb{N}_0^2$ is represented visually as follows:



Then $\alpha_n \in \mathrm{Aut}(\mathcal{M}, \psi)$ for all $n \in \mathbb{N}_0$ and it is easy to check that

$$\alpha_k \alpha_l = \alpha_{l+1} \alpha_k$$

for all $0 \leq k < l < \infty$, which are precisely the relations of F.
Hence defining $\tilde{\rho}(g_n) := \alpha_n$ gives a representation $\tilde{\rho} : F \to \mathrm{Aut}(\mathcal{M}, \psi)$.
Applying Theorem 1 to the representation $\tilde{\rho}$ gives the following:

$(\mathcal{M}, \psi, \alpha_0, \mathcal{M}^{\alpha_1})$ is a bilateral stationary noncommutative Markov process.

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Representations of F from Markov processes

We will now show a partial converse to Theorem 1 connected to the "illustrative example". To be able to use the tensor product construction done there, for a given transition operator R on a NCPS $(\mathcal{A}, \varphi)$ associated to a bilateral stationary noncommutative Markov process, we would first like to find a NCPS $(\mathcal{C}, \chi)$ and $\gamma \in \mathrm{Aut}(\mathcal{A} \otimes \mathcal{C}, \varphi \otimes \chi)$ such that

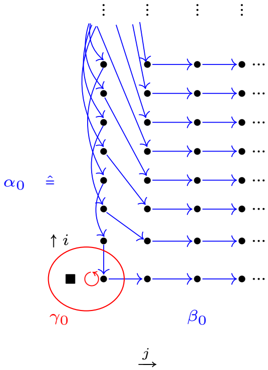
$$R = \iota_0^* \gamma \iota_0.$$

Here $\iota_0(a) := a \otimes \mathbb{1}_{\mathcal{C}}$.

A result of Kümmerer (1986) gives that if $\mathcal{A}$ is commutative with separable predual, then for $\mathcal{C} = \mathcal{L} := L^\infty([0,1],\lambda)$ and $\chi = \mathrm{tr}_\lambda := \int_{[0,1]} \cdot \, d\lambda$, there exists $\gamma \in \mathrm{Aut}(\mathcal{A} \otimes \mathcal{L}, \varphi \otimes \mathrm{tr}_\lambda)$ such that with $\iota_0(a) := a \otimes \mathbb{1}_{\mathcal{L}}$,

$$R = \iota_0^* \gamma \iota_0.$$

We will extend $\gamma \in \mathrm{Aut}(\mathcal{A} \otimes \mathcal{C}, \varphi \otimes \chi)$ to an automorphism $\gamma_0 \in \mathrm{Aut}(\mathcal{M}, \psi)$ as shown below (and seen before):



Theorem (Köstler and K. 2022)

Let $(\mathcal{A}, \varphi)$ *be a probability space where* $\mathcal{A}$ *is commutative with separable predual, and let* R *be a transition operator on* $\mathcal{A}$ *associated to a Markov process. Then there exists a probability space* $(\mathcal{M}, \psi)$*, representations* $\rho, \tilde{\rho} : F \to \mathrm{Aut}(\mathcal{M}, \psi)$*, and an embedding* $\iota : (\mathcal{A}, \varphi) \to (\mathcal{M}, \psi)$ *such that*

1. $\iota(\mathcal{A}) = \mathcal{M}^{\rho(g_0)}$,
2. $R^n = \iota^* \tilde{\rho}(g_0^n)\iota$ *for all* $n \in \mathbb{N}_0$.

Upshot: This result allows us to express a (classical) transition operator R as a compression of a represented generator of the Thompson group F.

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We obtained the following:

Representations of the Thompson group F

$\Downarrow$ Theorem 1

Bilateral noncommutative stationary Markov processes

Bilateral (classical) stationary Markov processes

$\Downarrow$ Theorem 2

Representations of the Thompson group F

Unilateral Markov processes and representations of F^+

Our approach is motivated by analogous results for F^+:

Representations of the monoid

$$F^+ = \langle g_0, g_1, \ldots \mid g_k g_l = g_{l+1} g_k, 0 \leq k < l < \infty \rangle_{\text{monoid}}$$

$\updownarrow$ partial spreadability

Unilateral noncommutative stationary Markov processes

C. Köstler, A. Krishnan, S. Wills (2020). Markovianity and the Thompson Monoid F^+. Preprint, arXiv:2009.14811.

This approach in turn was motivated by the action of the partial shifts monoid:

Representations of the partial shifts monoid

$$\mathcal{S} = \langle \theta_0, \theta_1, \ldots \mid \theta_k \theta_l = \theta_{l+1}\theta_k, 0 \le k \le l < \infty \rangle^+$$

↕ spreadability

Unilateral noncommutative Bernoulli shifts

D. G. Evans, R. Gohm, C. Köstler (2017). Semi-cosimplicial objects and spreadability. Rocky Mountain Journal of Mathematics, 47(6), 1839-1873.

- Commutative case: How do these results relate to work of P. Diaconis and D. Freedman on a de Finetti theorem for Markov chains? [1]

- Noncommutative case: Does Markovianity relate to results of A. Brothier and V. F. R. Jones on unitary "Pythagorean" representations of the Thompson group F? [2]

[1] P. Diaconis, D. Freedman (1980). De Finetti's theorem for Markov chains. The Annals of Probability, 115–130.

[2] A. Brothier, V .F. R. Jones (2019). Pythagorean representations of Thompson's groups. Journal of Functional Analysis, 277(7), 2442–2469.

Thank You!

A distributional invariance principle

Definition (Köstler, K., Wills 2020)

A sequence of random variables $\iota = (\iota_n)_{n\geq 0} : (\mathcal{A}, \varphi) \to (\mathcal{M}, \psi)$ is **partially spreadable** if there exists a representation $\rho : F^+ \to \mathrm{End}(\mathcal{M}, \psi)$ such that

$$\begin{aligned} \rho(g_0^n)\iota_0 &= \iota_n, \quad n \in \mathbb{N} && \text{(stationarity)}, \\ \iota_0(\mathcal{A}) &\subseteq \cap_{k\geq 1} \mathcal{M}^{\rho(g_k)} && \text{(localisation)}. \end{aligned}$$

Motivation: Replacing F^+ by $\mathcal{S} = F^+/\sim$ in the above definition gives an equivalent definition of **spreadability**.

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Definition

A sequence of random variables $\iota \equiv (\iota_n)_{n\geq 0} : (\mathcal{A}, \varphi) \to (\mathcal{M}, \psi)$ is said to be *maximal partially spreadable* if there exists a representation $\rho : F^+ \to \mathrm{End}(\mathcal{M}, \psi)$ such that

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We can now state the following de Finetti type result in the case of a classical probability space:

Theorem (Köstler, K., Wills, 2020)

Let $(\mathcal{A}, \varphi)$ *and* $(\mathcal{M}, \psi)$ *be probability spaces such that* $\mathcal{A}$ *and* $\mathcal{M}$ *are commutative with separable predual. Let* $\iota \equiv (\iota_n)_{n \in \mathbb{N}_0} : (\mathcal{A}, \varphi) \to (\mathcal{M}, \psi)$ *be a sequence of random variables. Then the following are equivalent:*

(a) ι *is a maximal partially spreadable sequence;*

(b) ι *is a stationary Markov sequence.*

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Theorem (Köstler, K., Wills 2020)

A maximal partially spreadable sequence ι is a stationary Markov sequence.

The converse result is more delicate ...

Theorem (Köstler, K., Wills 2020)

*A **"nice"** stationary Markov sequence ι is partially spreadable.*

Here "**nice**" means that the stationary Markov sequence can be produced as a so-called **coupling to a spreadable noncommutative Bernoulli shift**. Roughly speaking, the results as available in the context of tensor products of von Neumann algebras stay true "in spirit".

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