# Math 21C-2
# Practice Midterm

**Name:** Solutions

**Signature:** 

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- There are **ten** (plus cover and bonus) pages to the exam.
- The exam totals 100 points, plus 10 bonus points.
- You will have 90 minutes to complete the exam.
- No calculators, notes, or books allowed.
- Good luck!

| Problem | Points | Your Score |
|---|---|---|
| 1 | 10 | 10 |
| 2 | 10 | 10 |
| 3 | 10 | 10 |
| 4 | 10 | 10 |
| 5 | 10 | 10 |
| 6 | 10 | 10 |
| 7 | 10 | 10 |
| 8 | 10 | 10 |
| 9 | 10 | 10 |
| 10 | 10 | 10 |
| Bonus | 10 | 10 |
| Total | 100 | 110 |

**1.** (10 points) Definitions and Examples:

a. (2 points) Write the definition for a sequence to be *bounded above*. (An example is **not** sufficient for full credit.)

A sequence $\{a_n\}$ is bounded above if there exists an $M$ such that $a_n \leq M$ for all $n$.

b. (2 points) Write the definition of an *alternating series.* (An example is **not** sufficient for full credit.)

A series is an alternating series if the terms are alternately positive and negative.

c. (2 points) Write the definition for a sequence to *diverge to infinity.* (An example is **not** sufficient for full credit.)

$\{a_n\}$ diverges to infinity if for every $M$, there is an $N$ such that $n > N \Rightarrow a_n > M$.

d. (2 points) Write the definition of the *cross product* of two vectors. (An example is **not** sufficient for full credit.)

The cross product of $\vec{u}$ and $\vec{v}$ is

$$\vec{u} \times \vec{v} = (|\vec{u}||\vec{v}| \sin \theta)\vec{n},$$

where $\vec{n}$ is the unit vector perpendicular to the plane containing $\vec{u}$ and $\vec{v}$, and satisfying the Right-Hand Rule.

e. (2 points) Let $f = f(x)$. Write the definition of the *Maclaurin series* for $f$. (An example is **not** sufficient for full credit.)

The Maclaurin series for $f$ is $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

**2**. (10 points) Short Answers

a. (5 points) State the *Integral Test* for infinite series.

Let $\{a_n\}$ be a sequence of positive terms.

Suppose $f$ is a continuous, positive, decreasing function of $x$ for all $x \geq N$ ( $N$ a positive integer ), and $f(n) = a_n$ for all $n \geq N$.

Then $\sum_{n=N}^{\infty} a_n$ and $\int_N^{\infty} f(x)\,dx$ converge or diverge together.

b. (5 points) State the *nth-Term test* for infinite series.

If $\lim_{n\to\infty} a_n$ does not exist or is not equal to zero,

Then $\sum_{n=1}^{\infty} a_n$ diverges.

**3.** (10 points) Determine whether the following sequences converge or diverge. If a sequence converges, find its limit.

a. (3 points) $a_n = \ln(n+1) - \ln(n)$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln(n+1) - \ln(n) = \lim_{n\to\infty} \ln\left(\frac{n+1}{n}\right)$$

$$= \ln\left(\lim_{n\to\infty} \frac{n+1}{n}\right)$$

$$= \ln(1)$$

$$= 0$$

b. (3 points) $b_n = n2^{-\ln(n)}$.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} n2^{-\ln n} = \lim_{n\to\infty} e^{\ln(n 2^{-\ln n})} = \lim_{n\to\infty} e^{\ln n + \ln 2^{-\ln n}}$$

$$= \lim_{n\to\infty} \left(e^{\ln n}\right)\left(\frac{1}{e^{\ln(n)}}\right)^{\ln(2)}$$

$$= \lim_{n\to\infty} n\left(\frac{1}{n}\right)^{\ln(2)} = \lim_{n\to\infty} n^{1-\ln(2)} \to \infty \text{ since } \ln(2) < 1.$$

c. (4 points) $a_n = \dfrac{n!}{(-3)^n}$.

The limit does not exist.

While $\displaystyle\lim_{n\to\infty} \frac{n!}{3^n} = \infty$,

the limit $\displaystyle\lim_{n\to\infty} \frac{n!}{(-3)^n}$ alternates between being positively large, and negatively large.

**4.** (10 points) Determine whether the following series converge conditionally, converge absolutely, or diverge.

a. (5 points) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n}$.

Consider $\displaystyle\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{(\ln n)^n} \right| = \sum_{n=2}^{\infty} \frac{1}{\ln(n)^n}$.

We apply the Root Test to this series:

$$\sqrt[n]{\frac{1}{(\ln n)^n}} = \frac{1}{\ln n} \to 0 < 1.$$

So $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$ converges. And by the Absolute Convergence Test, $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n}$ converges (absolutely).

b. (5 points) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{0.999999}}$.

Consider $\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^{0.999999}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{0.999999}}$

this series diverges by the p-Series Test.

However, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{0.999999}}$ is an Alternating Series, with $u_n = \dfrac{1}{n^{0.999999}}$

$u_n > 0$ for all $n$

$u_n \geq u_{n+1}$ for all $n$

and $u_n \to 0$ as $n \to \infty$.



So by the Alternating Series Test,

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{0.999999}}$ converges (conditionally)

**5.** (10 points) Determine the values of $x$ for which the following series converges conditionally, converges absolutely, or diverges. What are the center, radius, and interval of convergence?

$$\sum_{n=2}^{\infty} n^n x^n.$$

We consider $\sum_{n=2}^{\infty} |n^n x^n|$, and apply the Root Test.

$\sqrt[n]{|n^n x^n|} = n|x| \longrightarrow \infty$ unless $x = 0$.

So the center of convergence is $x = 0$.

The radius of convergence is $R = 0$.

And the interval of convergence is $x = 0$.

Convergence is absolute at $x = 0$.

Notice that for $x \neq 0$, $\lim_{n\to\infty} n^n x^n$ is either $\infty$ $(x > 0)$ or does not exist $(x < 0)$.

So the series diverges for $x \neq 0$ by the nth Term Test.

The series is conditionally convergent nowhere.

**6**. (10 points) Compute (no shortcuts!!) the *Taylor series* centered at 1 for the function

$$f(x) = \sqrt{x}.$$

$f(x) = x^{\frac{1}{2}}$ $\qquad$ $f(1) = 1$

$f'(x) = \frac{1}{2} x^{-\frac{1}{2}}$ $\qquad$ $f'(1) = \frac{1}{2}$

$f''(x) = -\frac{1}{2^2} x^{-\frac{3}{2}}$ $\qquad$ $f''(1) = -\frac{1}{2^2}$

$f'''(x) = \frac{3 \cdot 1}{2^3} x^{-\frac{5}{2}}$ $\qquad$ $f'''(1) = \frac{3 \cdot 1}{2^3}$

$f^{(4)}(x) = -\frac{5 \cdot 3 \cdot 1}{2^4} x^{-\frac{7}{2}}$ $\qquad$ $f^{(4)}(1) = -\frac{5 \cdot 3 \cdot 1}{2^4}$

$\vdots$

$f^{(n)}(x) = (-1)^{n+1} \frac{(2n-3)(2n-5) \cdots 5 \cdot 3 \cdot 1}{2^n} x^{-\left(\frac{2n-1}{2}\right)}$ $\qquad$ $f^{(n)}(1) = (-1)^{n+1} \frac{(2n-3)(2n-5) \cdots 3 \cdot 1}{2^n}$

So the series is

$$1 + \frac{1}{2}(x-1) - \frac{1}{2^2 2!}(x-1)^2 + \frac{3 \cdot 1}{2^3 3!}(x-1)^3 - \cdots$$

$$= 1 + \frac{1}{2}(x-1) + \sum_{n \geq 2}^{\infty} \frac{(-1)^{n+1}(2n-3)(2n-5) \cdots 5 \cdot 3 \cdot 1}{2^n \, n!}(x-1)^n$$

**7.** (10 points) Using what you know about familiar Taylor series, write a power series for $f(x) = e^{-x^2}$. Use the first 3 non-zero terms to estimate $\int_0^1 e^{-x^2}\,dx$.

(Hint: you should be familiar with the series for $e^x$.)

Recall: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

So $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^2}{n!}$

$= 1 - x^2 + \frac{x^4}{2!} - \cdots$

$\int_0^1 e^{-x^2}\,dx \approx \int_0^1 1 - x^2 + \frac{x^4}{2}\,dx$

$= \left[ x - \frac{x^3}{3} + \frac{x^5}{10} \right]_0^1$

$= 1 - \frac{1}{3} + \frac{1}{10}$

$= \frac{23}{30}$

8. (10 points) Let $P$ be the point $(\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, -1)$, and $Q$ be the point $(-\frac{\sqrt{2}}{4}, -\frac{\sqrt{2}}{4}, -1)$.

a. (3 points) Find the component form for $\overrightarrow{PQ}$.

$$\overrightarrow{PQ} = \left\langle -\frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4}, -\frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4}, -1 - (-1) \right\rangle$$

$$= \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0 \right\rangle$$

b. (3 points) Find the magnitude $|\overrightarrow{PQ}|$.

$$|\overrightarrow{PQ}| = \sqrt{\left(-\frac{\sqrt{2}}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}\right)^2 + 0^2}$$

$$= \sqrt{\frac{2}{4} + \frac{2}{4} + 0}$$

$$= 1$$

c. (4 points) Find the unit vector in the direction of $\overrightarrow{PQ}$.

$\overrightarrow{PQ}$ is a unit vector, so $\overrightarrow{PQ}$ is the unit vector in the direction of $\overrightarrow{PQ}$.

**9.** (10 points) Let $\vec{u} = 2\vec{\imath} + 3\vec{\jmath} - \vec{k}$, and $\vec{v} = 3\vec{\imath} - 2\vec{\jmath} + 20\vec{k}$.

a. (3 points) Find $\vec{u} \cdot \vec{v}$.

$$\begin{aligned}(2\vec{\imath} + 3\vec{\jmath} - \vec{k}) \cdot (3\vec{\imath} - 2\vec{\jmath} + 20\vec{k}) &= (2)(3) + (3)(-2) + (-1)(20) \\ &= 6 - 6 - 20 \\ &= -20\end{aligned}$$

b. (3 points) Find $\vec{u} \times \vec{v}$.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2 & 3 & -1 \\ 3 & -2 & 20 \end{vmatrix} = \vec{\imath}\left(3(20) - (-1)(-2)\right) - \vec{\jmath}\left(2(20) - (-1)(3)\right) + \vec{k}\left(2(-2) - 3(3)\right)$$

$$= \vec{\imath}(60 - 2) - \vec{\jmath}(40 + 3) + \vec{k}(-4 - 9)$$

$$= 58\vec{\imath} - 43\vec{\jmath} - 13\vec{k}$$

c. (4 points) Find $\vec{u} \cdot (\vec{u} \times \vec{v})$.

$$\begin{aligned}\vec{u} \cdot (\vec{u} \times \vec{v}) &= (2\vec{\imath} + 3\vec{\jmath} - \vec{k}) \cdot (58\vec{\imath} - 43\vec{\jmath} - 13\vec{k}) \\ &= 2(58) + 3(-43) + (-1)(-13) \\ &= 0.\end{aligned}$$

Alternatively, $(\vec{u} \times \vec{v})$ produces a vector which is orthogonal to both $\vec{u}$ and $\vec{v}$.

So $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$,

by the Properties of the Dot Product, and the Definition of Orthogonality.

**10.** (10 points) Find parametric equations for the line which passes through $(2, 4, 5)$ and is perpendicular to the plane $3x + 7y - 5z = 21$.

Using the Equation for a Plane, we determine that

$\vec{n} = \langle 3, 7, -5 \rangle$ is a normal vector to the plane.

Then, we use the Parametric Equations for a Line

through $P_0(x_0, y_0, z_0)$ and parallel to $\langle v_1, v_2, v_3 \rangle$:

$x = x_0 + t v_1 \quad , \quad y = y_0 + t v_2 \quad , \quad z = z_0 + t v_3.$

So the Equations for a line through $(2,4,5)$, and

perpendicular to the plane (i.e. parallel to $\vec{n} = \langle 3, 7, -5 \rangle$) are

$x = 2 + t(3) \qquad y = 4 + t(7) \qquad z = 5 + t(-5)$

So $\quad x = 2 + 3t \qquad y = 4 + 7t \qquad z = 5 - 5t$

$-\infty < t < \infty.$

**Bonus.** (10 points) Let $p_n$ denote the nth prime: $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, $p_5 = 11$, etc. Determine whether the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{1}{p_n^2}$$

Notice that for all $n$, $p_n > n$.

So $\sum_{n=1}^{\infty} \frac{1}{p_n^2}$ converges by comparison to $\sum_{n=1}^{\infty} \frac{1}{n^2}$.