Math 21C-2

Practice Final Exam
July 31, 2008

Name: Solutions

Signature:______________________________

Student ID:______________________________

- There are **ten** (plus cover and bonus) pages to the exam.
- The exam totals 100 points, plus 10 bonus points.
- You will have 90 minutes to complete the exam.
- No calculators, notes, or books allowed.
- Good luck!

Problem	Points	Your Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Bonus	10	
Total	100	

1. (10 points) True or False. Remember, if a statement is not *always* true, then it is false. And if any part of the statement is false, the statement is false.

a. (2 points) If $a_n \geq 0$ for all n, $\sum_{n=0}^{\infty} c_n$ converges, and $a_n \geq c_n$ for all n, then $\sum_{n=0}^{\infty} a_n$ converges.

False. Being bigger than terms that have a convergent sum tells you nothing.

b. (2 points) A power series can only converge at one of the endpoints of the interval of convergence.

False. Consider $\sum_{n=0}^{\infty} \frac{x^n}{n^2}$. Interval of convergence is $[-1, 1]$.

$x=1$: $\sum_{n=0}^{\infty} \frac{1}{n^2}$ converges, $x=-1$: $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2}$ converges.

c. (2 points) If a function $f(x, y)$ has different limits along two different paths as (x, y) approaches (x_0, y_0), then $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist.

True.

d. (2 points) An absolute minimum of a function is also a local minimum.

True,

e. (2 points) If $f(x, y) = x^2 + y$, $x(t) = \cos(t)$, $y(t) = \sin(t)$, and $t(s) = \frac{1}{s}$, then $\frac{df}{ds} = -2\cos\left(\frac{1}{s}\right)\sin\left(\frac{1}{s}\right) + 2\cos\left(\frac{1}{s}\right)$.

False. $\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{dt}\underline{\frac{dt}{ds}} + \frac{\partial f}{\partial y}\frac{dy}{dt}\underline{\frac{dt}{ds}}$

2. (10 points) Prove that if $f(x) = \sum_{n=0}^{\infty} c_n x^n$, then $c_n = \frac{f^{(n)}(0)}{n!}$ for all n.

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$$

$$\boxed{f(0) = c_0}$$

$$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots$$

$$\boxed{f'(0) = c_1}$$

$$f''(x) = 2c_2 + 3(2)c_3 x + 4(3)c_4 x^2 + \cdots$$

$$\boxed{f''(0) = 2c_2}$$

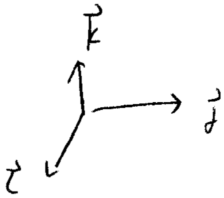
$$f'''(x) = 3(2)c_3 + 4(3)(2)c_4 x + \cdots$$

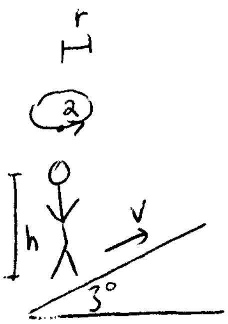
$$\boxed{f'''(0) = 3(2)c_3}$$

$$\vdots$$

$$\boxed{f^{(n)}(0) = n!\,c_n}$$

So $\boxed{c_n = \frac{f^{(n)}(0)}{n!}}$





4. a. (10 points) A Hiker

a. (4 points) A hiker is walking straight up a hill that is at a 3° incline. She is h meters tall, and walking with speed v meters per second. A tenacious mosquito is flying horizontal circles of radius r meters about her head, the whole time. He is traveling at constant speed 2 meter per second in relation to her head. Write a vector-valued function to describe the position of the mosquito in space.

The position of the hiker's head can be described by $\vec{r}(t) = h\vec{k} + (v\cos 3^\circ)t\vec{j} + (v\sin 3^\circ)t\vec{k}$

$$= (v\cos 3^\circ)t\,\vec{j} + (v\sin 3^\circ t + h)\vec{k}$$

So the mosquito's position can be described by

$$\vec{m}(t) = \vec{r}(t) + \left[(r\cos 2t)\vec{i} + (r\sin 2t)\vec{j}\right]$$

$$= (r\cos(2t))\vec{i} + \left((v\cos 3^\circ)t + (r\sin(2t))\right)\vec{j} + \left((v\sin 3^\circ)t + h\right)\vec{k}$$

b. (3 points) What is the mosquito's velocity (with respect to the ground)?

$$\frac{d\vec{m}}{dt} = (-2r\sin(2t))\vec{i} + \left(v\cos 3^\circ + 2r\cos(2t)\right)\vec{j} + (v\sin 3^\circ)\vec{k}$$

c. (3 points) Find an equation for the line tangent to the mosquito's path at time $t = \frac{\pi}{6}$.

$$\left.\frac{d\vec{m}}{dt}\right|_{t=\frac{\pi}{6}} = \left(-2r\sin\left(\frac{\pi}{3}\right)\right)\vec{i} + \left(v\cos 3^\circ + 2r\cos\left(\frac{\pi}{3}\right)\right)\vec{j} + (v\sin 3^\circ)\vec{k}$$

$$= (-r\sqrt{3})\vec{i} + (v\cos 3^\circ + r)\vec{j} + (v\sin 3^\circ)\vec{k}$$

$$\vec{m}\left(\frac{\pi}{6}\right) = \left(r\cos\left(\frac{\pi}{3}\right)\right)\vec{i} + \left((v\cos 3^\circ)\frac{\pi}{6} + r\sin\left(\frac{\pi}{3}\right)\right)\vec{j} + \left((v\sin 3^\circ)\frac{\pi}{6} + h\right)\vec{k}$$

$$= \left(\frac{r}{2}\right)\vec{i} + \left((v\cos 3^\circ)\frac{\pi}{6} + \frac{r\sqrt{3}}{2}\right)\vec{j} + \left((v\sin 3^\circ)\frac{\pi}{6} + h\right)\vec{k}$$

So the Tangent Line has Equation:

$$\boxed{x = \frac{r}{2} + t(-r\sqrt{3})}, \quad \boxed{y = \left((v\cos 3^\circ)\frac{\pi}{6} + \frac{r\sqrt{3}}{2}\right) + t(v\cos 3^\circ + r)}, \quad \boxed{z = \left(v\sin 3^\circ\frac{\pi}{6} + h\right) + t(v\sin 3^\circ)}$$

5. (10 points) Find the following limits, if they exist. If the limit does not exist, explain why not.

a. (5 points) $\displaystyle\lim_{\substack{(x,y)\to(2,2)\\ x+y\neq 4}} \frac{x+y-4}{\sqrt{x+y}-2}$.

$$\lim_{\substack{(x,y)\to(2,2)\\ x+y\neq 4}} \frac{x+y-4}{\sqrt{x+y}-2} = \lim_{\substack{(x,y)\to(2,2)\\ x+y\neq 4}} \frac{x+y-4}{\sqrt{x+y}-2}\cdot\frac{\sqrt{x+y}+2}{\sqrt{x+y}+2} = \lim_{\substack{(x,y)\to(2,2)\\ x+y\neq 4}} \frac{\cancel{(x+y-4)}(\sqrt{x+y}+2)}{\cancel{(x+y-4)}}$$

$$= \lim_{\substack{(x,y)\to(2,2)\\ x+y\neq 4}} \sqrt{x+y}+2 = \sqrt{2+2}+2 = 2+2 = 4$$

b. (5 points) $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ xy\neq 0}} \frac{|xy|}{xy}$.

The limit does not exist.

Consider the limit along $x=y$.

$$\lim_{\substack{(x,y)\to(0,0)\\ xy\neq 0\\ x=y}} \frac{|xy|}{xy} = \lim_{\substack{(x,y)\to(0,0)\\ xy\neq 0\\ x=y}} \frac{|x^2|}{x^2} = 1$$

But along $x=-y$

$$\lim_{\substack{(x,y)\to(0,0)\\ xy\neq 0\\ x=-y}} \frac{|xy|}{xy} = \lim_{\substack{(x,y)\to(0,0)\\ xy\neq 0\\ x=-y}} \frac{|-x^2|}{-x^2} = -1.$$

6. (10 points) Use any method to find the following derivatives or partial derivatives.

a. (3 points) $w = x^2 e^{2y}\cos(3z)$. Find $\dfrac{dw}{dt}$ at $(1, \ln 2, 0)$, on the curve $x = \cos(t)$, $y = \ln(t+2)$, $z = t$.

At $(1, \ln 2, 0)$, $z=0$, so $t=0$.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= \left(2x e^{2y}\cos 3z\right)\left(-\sin t\right) + \left(2x^2 e^{2y}\cos(3z)\right)\left(\frac{1}{t+2}\right) + \left(-3x^2 e^{2y}\sin 3z\right)(1)$$

$$\left.\frac{dw}{dt}\right|_{(1,\ln 2,0)} = \left(2(1)e^{2\ln(2)}\cos(3\cdot 0)\right)\left(\cancel{-\sin(0)}\right)^{\to 0} + \left(2(1)^2 e^{2\ln(2)}\cos(3\cdot 0)\right)\left(\frac{1}{0+2}\right) + \left(-3(1)^2 e^{2\ln(2)}\cancel{\sin(3\cdot 0)}\right)$$

$$= 2(4)\left(\tfrac{1}{2}\right) + (-3)(4)(0) = \boxed{4}$$

b. (3 points) $w = xy + \ln z$, $x = \dfrac{v^2}{u}$, $y = u + v$, $z = \cos(u)$. Find $\dfrac{\partial w}{\partial v}$ when $(u, v) = (-1, 2)$.

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} = (y)\left(\frac{2v}{u}\right) + (x)(1) + \left(\frac{1}{z}\right)(0) = y\left(\frac{2v}{u}\right) + x$$

When $(u,v) = (-1,2)$, $x = \frac{2^2}{-1} = -4$, $y = (-1+2) = 1$,

So $$\left.\frac{\partial w}{\partial v}\right|_{(u,v)=(-1,2)} = (1)\left(\frac{2(2)}{-1}\right) + (-4) = \boxed{-8}$$

c. (4 points) $g(x, y) = x^2 y + \cos(y) + y\sin(x)$. Find $\dfrac{\partial g}{\partial y\,\partial x\,\partial y}$.

$$\frac{\partial g}{\partial y\,\partial x\,\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(\frac{\partial g}{\partial y}\right)\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(x^2 - \sin y + \sin x\right)\right)$$

$$= \frac{\partial}{\partial y}\left(2x + \cos x\right)$$

$$= \boxed{0}$$

7. (10 points) Superman has been trapped by the evil Lex Luthor. There is a block of Kryptonite at the coordinates $(0,0,0)$, creating an energy field with strength given by the funcion $K(x,y,z) = \left(\frac{1}{\text{distance from the source}}\right)$. What Lex Luthor does not realize is that there is also an intense yellow light source at $(4,4,0)$, with intensity given by $L(x,y,z) = \left(\frac{1}{\text{distance from the source}}\right)$. So Superman's strength is given by $S(x,y,z) = L(x,y,z) - K(x,y,z)$. If Superman is at the coordinates $(5,8,2)$, then in what direction (unit vector) should Superman fly if he wants to increase his strength most rapidly?

We seek the direction of the gradient $\nabla S|_{(5,8,2)}$.

$$\nabla S = \langle S_x, S_y, S_z \rangle = \left\langle \frac{\partial}{\partial x}\left[\frac{1}{\sqrt{(x-4)^2+(y-4)^2+z^2}} - \frac{1}{\sqrt{x^2+y^2+z^2}}\right], \frac{\partial}{\partial y}\left[L-K\right], \frac{\partial}{\partial z}\left[L-K\right] \right\rangle$$

$$= \left\langle \frac{-2(x-4)}{2\left((x-4)^2+(y-4)^2+z^2\right)^{3/2}} + \frac{2x}{2(x^2+y^2+z^2)^{3/2}}, \frac{-2(y-4)}{2\left((x-4)^2+(y-4)^2+z^2\right)^{3/2}} + \frac{2y}{2(x^2+y^2+z^2)^{3/2}}, \right.$$

$$\left. \frac{-2z}{2\left((x-4)^2+(y-4)^2+z^2\right)^{3/2}} + \frac{2z}{2\left((x)^2+y^2+z^2\right)^{3/2}} \right\rangle$$

So
$$\nabla S|_{(5,8,2)} = \left\langle \frac{-(5-4)}{\left((5-4)^2+(8-4)^2+2^2\right)^{3/2}} + \frac{10}{(5^2+8^2+2^2)^{3/2}}, \frac{-(8-4)}{\left((5-4)^2+(8-4)^2+2^2\right)^{3/2}} + \frac{16}{(5^2+8^2+2^2)^{3/2}}, 0 \right\rangle$$

$$= \left\langle \frac{-1}{(1+16+4)^{3/2}} + \frac{10}{(25+64+4)^{3/2}}, \frac{-4}{(1+16+4)^{3/2}} + \frac{16}{(25+64+4)^{3/2}}, 0 \right\rangle$$

$$= \left\langle \frac{-1}{21^{3/2}} + \frac{10}{93^{3/2}}, \frac{-4}{21^{3/2}} + \frac{16}{93^{3/2}}, 0 \right\rangle$$

And clearly it is unreasonable to Calculate:

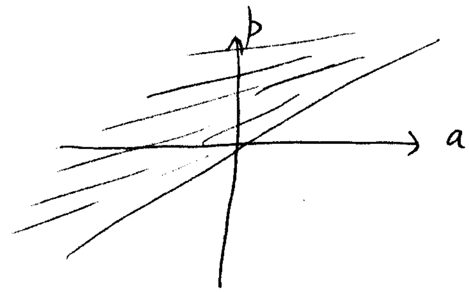
$$\frac{\nabla S|_{(5,8,2)}}{\left|\nabla S|_{(5,8,2)}\right|}$$

8. (10 points) Find two numbers a and b with $a \leq b$ such that

$$\int_a^b (6 - x + x^2)\, dx$$

has its largest value.

Let $f(a,b) = \int_a^b (6 - x + x^2)\, dx$. We want the maximal value of f over the domain $\{(a,b) \mid a \leq b\}$



$$f(a,b) = \int_a^b (6 - x + x^2)\, dx = 6x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 \Big]_a^b$$

$$= \left(6b - \tfrac{1}{2}b^2 + \tfrac{1}{3}b^3\right) - \left(6a - \tfrac{1}{2}a^2 + \tfrac{1}{3}a^3\right)$$

$\frac{\partial f}{\partial a} = -6 + a - a^2$, $\frac{\partial f}{\partial b} = 6 - b + b^2$ exist everywhere.

So the only critical points are where $\frac{\partial f}{\partial a} = \frac{\partial f}{\partial b} = 0$.

$-6 + a - a^2 = 0 \Rightarrow$ No solution.

There are no critical values on the interior of this domain.

At every point on the boundary, $a = b$, $f(a,a) = 0$.

So there is no choice for a, b which maximize $\int_a^b 6 - x + x^2\, dx$.

9. (10 points) Find all the relative maxima, relative minima, and saddle points of the function:

$$f(x, y) = \frac{1}{x} + xy + \frac{1}{y}$$

$\frac{\partial f}{\partial x} = -\frac{1}{x^2} + y$ $\qquad$ $\frac{\partial f}{\partial y} = x - \frac{1}{y^2}$ $\qquad$ Exist as long as $x \neq 0$, $y \neq 0$. But these points are not in the domain of the function anyway.

So the only critical points in the domain are where $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0.$

$-\frac{1}{x^2} + y = 0 \quad \Rightarrow \quad y = \frac{1}{x^2}$

$x - \frac{1}{y^2} = 0 \quad \Rightarrow \quad x - \frac{1}{\left(\frac{1}{x^2}\right)^2} = 0$

$x - x^4 = 0 \qquad x(1 - x^3) = 0$

So $\cancel{x = 0}$ or $x = 1$

But x cannot be 0 (Not in domain)

$x = 1 \Rightarrow y = \frac{1}{1^2} = 1$.

So the only critical pt. is $(1, 1)$.

$\frac{\partial^2 f}{\partial x^2} = \frac{2}{x^3}$, $\quad \frac{\partial^2 f}{\partial y^2} = \frac{2}{y^3}$, $\quad \frac{\partial^2 f}{\partial x \partial y} = 1.$

So at $(1,1)$, $\quad \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = \left(\frac{2}{1^3}\right)\left(\frac{2}{1^3}\right) - (1)^2 = 4 - 1 = 3 > 0.$

And $\frac{\partial^2 f}{\partial x^2} = \frac{2}{1^3} = 2 > 0,$

So f has a Local Mininimum at $(1,1)$ of

$f(1,1) = \frac{1}{1} + (1)(1) + \frac{1}{1} = 3$

10. (10 points) Maximize the function $f(x, y, z) = x^2 + 2y - z^2$ subject to the constraints $2x - y = 0$ and $y + z = 0$.

$g_1(x,y,z) = 2x - y \quad , \quad g_2(x,y,z) = y + z$

We want x, y, z, λ, μ so that

$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \quad , \quad g_1 = 0 \quad , \quad g_2 = 0$

$\searrow \langle 2x, 2, -2z \rangle = \lambda \langle 2, -1, 0 \rangle + \mu \langle 0, 1, 1 \rangle$

So we want $2x = 2\lambda \quad , \quad 2 = -\lambda + \mu \quad , \quad -2z = \mu ,$

$2x - y = 0 \quad , \quad$ and $\quad y + z = 0 .$

$2x = 2\lambda \Rightarrow x = \lambda$, and $-2z = \mu$

So $2 = -\lambda + \mu \Rightarrow 2 = -x + (-2z)$

$x = -2 - 2z$

Then, $2(-2-2z) - y = 0 \Rightarrow y = -4 - 4z ,$

So $(-4-4z) + z = 0 \Rightarrow -4 = 3z \Rightarrow \boxed{z = -\frac{4}{3}}$

Then $y = -4 - 4(-\frac{4}{3}) = -\frac{12}{3} + \frac{16}{3} = \boxed{\frac{4}{3} = y}$

And $x = -2 - 2(-\frac{4}{3}) = \frac{-6}{3} + \frac{8}{3} = \boxed{\frac{2}{3} = x}$

So the only critical point is $(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3})$,

Where, $f(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}) = (\frac{2}{3})^2 + 2(\frac{4}{3}) - (-\frac{4}{3})^2 = \frac{4}{9} + \frac{8}{3} - \frac{16}{9} = \frac{4}{3}$

Clearly $(0,0,0)$ is part of our constrained domain, an $f(0,0,0) = 0$, so $\frac{4}{3}$ must be a Maximum.

Bonus. (10 points) Find a solution to the initial value problem

$$y' - y = x \qquad y(0) = 1$$

by assuming that there is a solution of the form

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots.$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} + \cdots$$

Then $y' - y = (a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \cdots + (n a_n - a_{n-1})x^{n-1} + \cdots$

If this is to satisfy $y' - y = x$, then

$$a_1 - a_0 = 0$$
$$2a_2 - a_1 = 1$$
$$3a_3 - a_2 = 0$$
$$\vdots$$
$$n a_n - a_{n-1} = 0$$

Further, $y(0) = 1$, so

$$a_0 + a_1(0) + a_2(0)^2 + \cdots = 1$$
$$\Rightarrow \boxed{a_0 = 1}$$

So $\boxed{a_1 = 1}$,

$$2a_2 - 1 = 1 \Rightarrow \boxed{a_2 = \frac{2}{2} = 1}$$
$$3a_3 - a_2 = 0 \Rightarrow \boxed{a_3 = \frac{1}{3}}$$
$$4a_4 - a_3 = 0 \Rightarrow \boxed{a_4 = \frac{1}{3 \cdot 4}}$$
$$\vdots$$
$$n a_n - a_{n-1} = 0 \Rightarrow \boxed{a_n = \frac{2}{n!}}$$

Recall:
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

So $y = 1 + x + 2\frac{x^2}{2!} + 2\frac{x^3}{3!} + \frac{2x^4}{4!} + \cdots$

$$= 1 + x + 2(e^x - 1 - x)$$

$$\boxed{y = 2e^x - x - 1}$$