Math 21C-2
Midterm
July 10, 2008

Name: Key

Signature: ______________________

Student ID: ______________________

- There are **ten** (plus cover and bonus) pages to the exam.
- The exam totals 100 points, plus 10 bonus points.
- You will have 90 minutes to complete the exam.
- No calculators, notes, or books allowed.
- Good luck!

Problem	Points	Your Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Bonus	10	
Total	100	

1. (10 points) Write the definitions of the following terms. (An example is **not** sufficient for full credit.)

a. (2 points) A *nondecreasing sequence*

A sequence $\{a_n\}$ is called nondecreasing if $a_n \leq a_{n+1}$ for all n.

b. (2 points) A *geometric series*

A geometric series is a series of the form $\sum_{n=1}^{\infty} ar^{n-1}$

c. (2 points) A *unit vector*

A unit vector is a vector of magnitude 1.

d. (2 points) The *dot product* of two vectors

If $\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{v} = \langle v_1, v_2, v_3 \rangle$,

Then the dot product is $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$

e. (2 points) The *Taylor series* generated by $f(x)$ centered at 0

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

2. (10 points) Theorems

a. (5 points) State the *Root Test* for infinite series.

Let $\sum a_n$ be a series with $a_n \geq 0$ for $n \geq N$, and

suppose $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho$.

Then

(a) If $\rho < 1$, the series converges

(b) If $\rho > 1$ or ρ is infinite, the series diverges

(c) The test is inconclusive if $\rho = 1$.

b. (5 points) State the *Limit Comparison Test* for infinite series.

Suppose $a_n > 0$ and $b_n > 0$ for all $n \geq N$.

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ converge or diverge together.

2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, and $\sum b_n$ converges, then $\sum a_n$ converges

3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, and $\sum b_n$ diverges, then $\sum a_n$ diverges.

3. (10 points) Determine whether the following sequences converge or diverge. If a sequence converges, find its limit.

a. (3 points) $a_n = \dfrac{n^2}{2n-1}\sin\left(\dfrac{1}{n}\right)$.

$$\lim_{n\to\infty} \frac{n^2}{2n-1}\sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{2n-1}{n^2}\right)} \overset{L'H}{=} \lim_{n\to\infty} \frac{\cos\left(\frac{1}{n}\right)\left(-\frac{1}{n^2}\right)}{\left(\frac{n^2(2)-(2n-1)(2n)}{n^4}\right)}$$

$$= \lim_{n\to\infty} \frac{\cos\left(\frac{1}{n}\right)\left(-\frac{1}{n^2}\right)}{\left(\frac{-2n^2+2n}{n^4}\right)} = \lim_{n\to\infty} \frac{-\cos\left(\frac{1}{n}\right) n^4}{n^2(-2n^2+2n)} = \lim_{n\to\infty} \frac{-\cos\left(\frac{1}{n}\right) n}{-2n+2}$$

$$= \left(\lim_{n\to\infty} -\cos\left(\frac{1}{n}\right)\right)\left(\lim_{n\to\infty} \frac{n}{-2n+2}\right) = (-1)\left(-\frac{1}{2}\right) = \boxed{\frac{1}{2}}$$

b. (3 points) $b_n = \sin(\tan(e^{-n(\cos(n)+3)}))$.

$$\lim_{n\to\infty} \sin\left(\tan\left(e^{-n(\cos(n)+3)}\right)\right) = \sin\left(\tan\left(e^{\lim_{n\to\infty} -n(\cos(n)+3)}\right)\right)$$

$$= \sin(\tan(0))$$

$$= \sin(0)$$

$$= \boxed{0.}$$

c. (4 points) $a_n = \dfrac{(2n+1)!}{(3n^2-4)(2n-1)!}$.

$$\lim_{n\to\infty} \frac{(2n+1)!}{(3n^2-4)(2n-1)!} = \lim_{n\to\infty} \frac{(2n+1)\,2n\,\cancel{(2n-1)!}}{(3n^2-4)\cancel{(2n-1)!}}$$

$$= \lim_{n\to\infty} \frac{4n^2+2n}{3n^2-4} = \boxed{\frac{4}{3}}$$

4. (10 points) Determine whether the following series converge conditionally, converge absolutely, or diverge.

a. (5 points) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(n+1)(n+2)}{n!}$.

We use the Ratio Test on the series $\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^n (n+1)(n+2)}{n!} \right|$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+2)(n+3)}{(n+1)!} \cdot \frac{n!}{(n+1)(n+2)} \right| = \frac{(n+2)(n+3)}{(n+1)(n+2)} \cdot \frac{1}{n+1} \cdot \frac{\cancel{n!}}{\cancel{n!}} \to 0 < 1$$

So the series converges absolutely.

b. (5 points) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sqrt[n]{n}}$.

Let $u_n = \dfrac{1}{n\sqrt[n]{n}}$. $u_n > 0$ for all n.

u_n is decreasing

and $u_n \to 0$.

So the series converges by the Alternating Series Test.

Now, consider $\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n\sqrt[n]{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$.

We use Limit Comparison with $\frac{1}{n}$:

$$\lim_{n\to\infty} \frac{\frac{1}{n\sqrt[n]{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n\sqrt[n]{n}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1 > 0.$$

So by the LCT, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ converge or diverge together.

But we know $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (Harmonic Series),

So $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$ diverges.

Thus, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sqrt[n]{n}}$ converges conditionally.

5. (10 points) Determine the values of x for which the following series converges conditionally, converges absolutely, or diverges. What are the center, radius, and interval of convergence?

$$\sum_{n=0}^{\infty} \frac{(x+1)^{n+1}}{(n+2)4^{n-1}}.$$

We use the Ratio Test on the series $\displaystyle\sum_{n=0}^{\infty} \left| \frac{(x+1)^{n+1}}{(n+2)\,4^{n-1}} \right|$:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{(x+1)^{(n+1)+1}}{((n+1)+2)\,4^{(n+1)-1}} \cdot \frac{(n+2)\,4^{n-1}}{(x+1)^{n+1}} \right|$$

$$= \left| \frac{(x+1)^{n+2}\,(n+2)\,4^{n-1}}{(n+3)\,4^{n}\,(x+1)^{n+1}} \right|$$

$$= |x+1| \left(\frac{n+2}{n+3}\right) \frac{1}{4} \xrightarrow{\text{as } n\to\infty} \frac{1}{4}|x+1|$$

So the series converges (absolutely) when $\frac{1}{4}|x+1| < 1$

$$|x+1| < 4$$

$$-5 < x < 3$$

So the center is $x = -1$

The Radius of convergence is 4

To find the interval, we must check the endpoints.

For $x = 3$, the series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{4^{n+1}}{(n+2)\,4^{n-1}} = \sum_{n=0}^{\infty} \frac{4^2}{n+2}$ which diverges (by Limit Comparison to $\frac{1}{n}$)

For $x = -5$, the series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{(-4)^{n+1}}{(n+2)\,4^{n-1}} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\,4^2}{(n+2)}$ which converges (conditionally) by the Alternating Series Test.

So the interval of convergence is $-5 \le x < 3$

Absolute convergence $-5 < x < 3$

Conditional convergence $x = -5$

Divergence for all other x (by nth Term Test)

6. (10 points) Compute (no shortcuts!!) the *Taylor series* centered at 0 for the function

$$f(x) = e^{1-\frac{2x}{3}}.$$

$f(x) = e^{1-\frac{2x}{3}}$

$f(0) = e$

$f'(x) = -\frac{2}{3} e^{1-\frac{2}{3}x}$

$f'(0) = -\frac{2}{3}e$

$f''(x) = \frac{2}{3} \cdot \frac{2}{3} e^{1-\frac{2}{3}x}$

$f''(0) = \frac{4}{9} e$

$f'''(x) = -\frac{2^3}{3^3} e^{1-\frac{2}{3}x}$

$f'''(0) = -\frac{8}{27} e$

$\vdots$

$f^{(n)}(x) = (-1)^n \frac{2^n}{3^n} e^{1-\frac{2}{3}x}$

$f^{(n)}(0) = (-1)^n \frac{2^n}{3^n} e$

So the Taylor series is

$$e - \frac{2}{3} e x + \frac{4}{9 \cdot 2!} e x^2 - \frac{8}{27 \cdot 3!} e x^3 + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \left(\frac{2}{3}\right)^n \frac{e}{n!} x^n$$

7. (10 points) Assume that

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \text{ for } |x| \le 1.$$

a. (3 points) Use the 5th degree Taylor Polynomial to find an approximation for $\frac{\pi}{4}$.

$$P_5(x) = x - \frac{x^3}{3} + \frac{x^5}{5}$$

$$\frac{\pi}{4} = \tan^{-1}(1).$$

$$P_5(1) = 1 - \frac{1^3}{3} + \frac{1^5}{5} = \frac{15}{15} - \frac{5}{15} + \frac{3}{15} = \frac{13}{15} \approx \frac{\pi}{4}$$

b. (3 points) Estimate the error of the approximation from part a, given that $-720 \le f^{(7)}(t) \le 384$ for $0 \le t \le 1$, where $f(t) = \tan^{-1}(t)$.

First, we notice that $P_5(x) = P_6(x)$.

So by Taylor's Remainder Formula,

$$|R_5(1)| = |R_6(1)| \le M \frac{|1|^7}{7!}.$$ We take $M = 720$.

So $$|R_5(1)| \le \frac{720}{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1} = \frac{144}{7\cdot6\cdot4\cdot3\cdot2\cdot1} = \frac{12}{7\cdot6\cdot2\cdot1} = \frac{1}{7}$$

Alternatively, by the Alternating Series Estimate Theorem,

$|\text{Error}| \le \frac{1}{7}$,

The next nonzero term in the series.

c. (4 points) Explain why this method cannot be used to estimate $\frac{\pi}{3}$.

$$\frac{\pi}{3} = \tan^{-1}(\sqrt{3}).$$

But $\sqrt{3} > 1$, so $x = \sqrt{3}$ is outside the Interval of convergence for the Maclaurin Series.

8. (10 points) Let P be the point $(5, 7, -1)$, and Q be the point $(2, 9, -2)$.

a. (3 points) Find the component form for $\overrightarrow{PQ}$.

$$\overrightarrow{PQ} = \langle 2-5,\ 9-7,\ -2-(-1) \rangle$$

$$= \langle -3,\ 2,\ -1 \rangle$$

b. (3 points) Find the magnitude $|\overrightarrow{PQ}|$.

$$|\overrightarrow{PQ}| = \sqrt{(-3)^2 + (2)^2 + (-1)^2}$$

$$= \sqrt{9+4+1}$$

$$= \sqrt{14}$$

c. (4 points) Find the unit vector in the direction of $\overrightarrow{PQ}$.

$$\frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \frac{1}{\sqrt{14}} \langle -3, 2, -1 \rangle = \left\langle -\frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, -\frac{1}{\sqrt{14}} \right\rangle$$

$$= \left\langle -\frac{3\sqrt{14}}{14}, \frac{\sqrt{14}}{7}, -\frac{\sqrt{14}}{14} \right\rangle$$

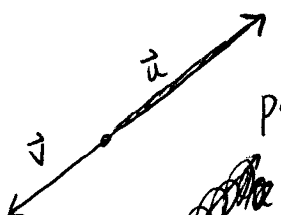
9. (10 points) Let $\vec{u} = 2\vec{i} - 2\vec{j} + 4\vec{k}$, and $\vec{v} = -\vec{i} + \vec{j} - 2\vec{k}$.

a. (3 points) Find $\vec{u} \times \vec{v}$.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & 4 \\ -1 & 1 & -2 \end{vmatrix} = \vec{i}(4-4) - \vec{j}(-4+4) + \vec{k}(2-2)$$

$$= \vec{0}$$

b. (3 points) Find the projection of $\vec{u}$ onto $\vec{v}$, $\text{proj}_{\vec{v}}\vec{u}$.



$$\text{proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}$$

$$= \frac{-12}{6}(-\vec{i} + \vec{j} - 2\vec{k})$$

$$= 2\vec{i} - 2\vec{j} + 4\vec{k}$$

$$= \vec{u}$$

$$\vec{u}\cdot\vec{v} = 2(-1) + (-2)(1) + 4(-2)$$

$$= -2 - 2 - 8$$

$$= -12$$

$$|\vec{v}|^2 = ((-1)^2 + (1)^2 + (-2)^2)$$

$$= 1 + 1 + 4$$

$$= 6$$

c. (4 points) Find $(proj_{\vec{v}}\vec{u}) \cdot (\vec{v} \times \vec{u})$.

$$\vec{u} \cdot \vec{0} = 0$$

10. (10 points) Find an equation for the plane through the points $(2,4,5)$, $(1,5,7)$, and $(-1,6,8)$.

Let's name the points: $P = (2,4,5)$

$Q = (1,5,7)$

(other choices of vector are okay) $R = (-1,6,8)$.

Then $\overrightarrow{QP} = \langle 2-1, 4-5, 5-7 \rangle = \langle 1, -1, -2 \rangle$

and $\overrightarrow{QR} = \langle -1-1, 6-5, 8-7 \rangle = \langle -2, 1, 1 \rangle$

Then $\vec{n} = \overrightarrow{QP} \times \overrightarrow{QR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & -2 \\ -2 & 1 & 1 \end{vmatrix} = \vec{i}(-1+2) - \vec{j}(1-4) + \vec{k}(1-2)$

$= \langle 1, 3, -1 \rangle$ is a vector normal to the plane containing $\overrightarrow{QP}$ and $\overrightarrow{QR}$.

(other choices of point are okay)

$P = (2,4,5)$ is a point in the plane.

So we use the Form for an Equation of a plane, with

$\vec{n} = \langle 1, 3, -1 \rangle$, $P = (2,4,5)$:

$$1(x-2) + 3(y-4) - (z-5) = 0$$

$$x - 2 + 3y - 12 - z + 5 = 0$$

$$x + 3y - z = 9$$

Bonus. (10 points) Prove that if $a_n > 0$, and $\lim_{n\to\infty} na_n \neq 0$, then $\sum_{n=0}^{\infty} a_n$ is divergent.

Use Limit Comparison with $\frac{1}{n}$.

$$\lim_{n\to\infty} \frac{a_n}{\frac{1}{n}} = \lim_{n\to\infty} n a_n \neq 0$$

So either

① $\lim_{n\to\infty} n a_n = c > 0.$

Then LCT says $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} \frac{1}{n}$ diverge together.

or ② $\lim_{n\to\infty} n a_n = \infty$

Then by LCT, $\sum_{n=0}^{\infty} \frac{1}{n}$ diverges $\Rightarrow \sum_{n=0}^{\infty} a_n$ diverges