Math 21C-2

Practice Final Exam
July 31, 2008

Name:____

Signature:_____________________________________

Student ID:___________________________________

- There are **ten** (plus cover and bonus) pages to the exam.
- The exam totals 100 points, plus 10 bonus points.
- You will have 90 minutes to complete the exam.
- No calculators, notes, or books allowed.
- Good luck!

Problem	Points	Your Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Bonus	10	
Total	100	

1. (10 points) True or False. Remember, if a statement is not *always* true, then it is false. And if any part of the statement is false, the statement is false.

a. (2 points) If $a_n \geq 0$ for all n, $\sum_{n=0}^{\infty} c_n$ converges, and $a_n \geq c_n$ for all n, then $\sum_{n=0}^{\infty} a_n$ converges.

b. (2 points) A power series can only converge at one of the endpoints of the interval of convergence.

c. (2 points) If a function $f(x, y)$ has different limits along two different paths as (x, y) approaches (x_0, y_0), then $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist.

d. (2 points) An absolute minimum of a function is also a local minimum.

e. (2 points) If $f(x, y) = x^2 + y$, $x(t) = \cos(t)$, $y(t) = \sin(t)$, and $t(s) = \frac{1}{s}$, then $\frac{df}{ds} = -2\cos\left(\frac{1}{s}\right)\sin\left(\frac{1}{s}\right) + 2\cos\left(\frac{1}{s}\right)$.

2. (10 points) Prove that if $f(x) = \sum_{n=0}^{\infty} c_n x^n$, then $c_n = \frac{f^{(n)}(0)}{n!}$ for all n.

3. (10 points) Vectors

a. (5 points) Prove that squares are the only rectangles with perpendicular diagonals.

b. (5 points) If $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$, and $\vec{a} \neq \vec{0}$, prove or disprove: $\vec{b} = \vec{c}$.

4. a. (10 points) A Hiker

a. (4 points) A hiker is walking straight up a hill that is at a 3° incline. She is h meters tall, and walking with speed v meters per second. A tenacious mosquito is flying horizontal circles of radius r meters about her head, the whole time. He is traveling at constant speed 2 meter per second in relation to her head. Write a vector-valued function to describe the position of the mosquito in space.

b. (3 points) What is the mosquito's velocity (with respect to the ground)?

c. (3 points) Find an equation for the line tangent to the mosquito's path at time $t = \frac{\pi}{6}$.

5. (10 points) Find the following limits, if they exist. If the limit does not exist, explain why not.

a. (5 points) $$\lim_{\substack{(x,y)\to(2,2) \\ x+y\neq 4}} \frac{x+y-4}{\sqrt{x+y}-2}.$$

b. (5 points) $$\lim_{\substack{(x,y)\to(0,0) \\ xy\neq 0}} \frac{|xy|}{xy}.$$

6. (10 points) Use any method to find the following derivatives or partial derivatives.

a. (3 points) $w = x^2 e^{2y} \cos(3z)$. Find $\dfrac{dw}{dt}$ at $(1, \ln 2, 0)$, on the curve $x = \cos(t)$, $y = \ln(t+2)$, $z = t$.

b. (3 points) $w = xy + \ln z$, $x = \dfrac{v^2}{u}$, $y = u + v$, $z = \cos(u)$. Find $\dfrac{\partial w}{\partial v}$ when $(u, v) = (-1, 2)$.

c. (4 points) $g(x, y) = x^2 y + \cos(y) + y \sin(x)$. Find $\dfrac{\partial g}{\partial y \partial x \partial y}$.

7. (10 points) Superman has been trapped by the evil Lex Luthor. There is a block of Kryptonite at the coordinates $(0, 0, 0)$, creating an energy field with strength given by the funcion $K(x, y, z) = \left(\frac{1}{\text{distance from the source}}\right)$. What Lex Luthor does not realize is that there is also an intense yellow light source at $(4, 4, 0)$, with intensity given by $L(x, y, z) = \left(\frac{1}{\text{distance from the source}}\right)$. So Superman's strength is given by $S(x, y, z) = L(x, y, z) - K(x, y, z)$. If Superman is at the coordinates $(5, 8, 2)$, then in what direction (unit vector) should Superman fly if he wants to increase his strength most rapidly?

8. (10 points) Find two numbers a and b with $a \leq b$ such that

$$\int_a^b (6 - x + x^2)\, dx$$

has its largest value.

9. (10 points) Find all the relative maxima, relative minima, and saddle points of the function:

$$f(x,y) = \frac{1}{x} + xy + \frac{1}{y}$$

10. (10 points) Maximize the function $f(x, y, z) = x^2 + 2y - z^2$ subject to the constraints $2x - y = 0$ and $y + z = 0$.

Bonus. (10 points) Find a solution to the initial value problem

$$y' - y = x \qquad y(0) = 1$$

by assuming that there is a solution of the form

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$