Math 108 - Section 1, Summer Session I, 2010
Quiz 1

Name: Key

1. State the definition of each of the following terms:

a. [2 pts.] A **contradiction**.

A contradiction is a propositional form that is false for every assignment of truth values to its components.

b. [3 pts.] For propositions P and Q, the **conditional sentence** $P \Rightarrow Q$.

The conditional sentence $P \Rightarrow Q$ is the proposition "If P, then Q." P is called the antecedent and Q is called the consequent. $P \Rightarrow Q$ is true if and only if P is false or Q is true.

2. Determine whether each of the following is a tautology, a contradiction, or neither. Justify your response.

a. [5 pts.] $P \iff P \wedge (P \vee Q)$.

P	Q	$P \vee Q$	$P \wedge (P \vee Q)$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	F

Tautology

b. [5 pts.] $P \wedge (P \iff Q) \wedge \sim Q$.

P	Q	$P \Leftrightarrow Q$	$\sim Q$	$P \wedge (P \Leftrightarrow Q) \wedge \sim Q$
T	T	T	F	F
T	F	F	T	F
F	T	F	F	F
F	F	T	T	F

Contradiction

3. [5 pts.] Give an English translation for each. The universe is given in parentheses.

a. $(\forall x)(x \geq 1)$. (Natural numbers)

All natural numbers are greater than or equal to one.

b. $(\forall x)(x \text{ is prime} \wedge x \neq 2 \Rightarrow x \text{ is odd})$. (Natural numbers)

Every natural number that is prime and different from 2 is odd.

c. $\sim (\exists x)(x^2 < 0)$. (Real numbers)

It is not the case that there is a real number whose square is less than zero.

OR

There is no real number whose square is negative.

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4. [20 pts.] Prove that for every real number $\epsilon > 0$, there is a natural number M such that if $m > n > M$, then $\frac{1}{n} - \frac{1}{m} < \epsilon$.

Let $\varepsilon > 0$. Now, let M be the integer part of $\frac{1}{\varepsilon}$, plus 1.

Then M is a natural number, since M is a positive integer, and $M > \frac{1}{\varepsilon}$.

Now, for all natural numbers $n > M$, $n > \frac{1}{\varepsilon} \Rightarrow \frac{1}{n} < \varepsilon$.

And, for all natural numbers m and n, with $m > n > M$,

$$\frac{1}{n} - \frac{1}{m} < \frac{1}{n} < \varepsilon.$$

$\therefore$ For every real number $\varepsilon > 0$, there is a natural number M s.t. $m > n > M \Rightarrow \frac{1}{n} - \frac{1}{m} < \varepsilon$.

5. [20 pts.] Prove that for every rational number z and every irrational number x, there exists a unique irrational number y such that $x + y = z$.

Let z be a rational number and x be an irrational number.

Then $-x$ is irrational. Let $y = z - x$.

Then $x + y = x + (z - x) = z$.

And $y = z - x$ is irrational because if we supposed that it were,

then $-y + z = (x - z) + z = x$ would be rational,

which is a contradiction.

$\therefore$ There exists an irrational number y s.t. $x + y = z$.

Now, suppose $\exists\, y_1, y_2$ s.t. $x + y_1 = z$ and $x + y_2 = z$.

Then $x + y_1 = x + y_2 \Rightarrow y_1 = y_2$.

$\therefore$ There exists a <u>unique</u> irrational number y s.t. $x + y = z$.

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6. a. [7 pts.] For sets A, B, and C, prove that if $A \subset B$, then $A \cap C \subset B \cap C$.

Assume $A \subseteq B$. Suppose $x \in A \cap C$. Then $x \in A$ and $x \in C$.

Since $x \in A$ and $A \subset B$, $x \in B$.

Thus $x \in B$ and $x \in C$, so $x \in B \cap C$.

$\therefore$ $A \cap C \subseteq B \cap C$.

b. [3 pts.] Show the converse is false.

We provide a counterexample.

Let $A = \{1\}$, $B = \{2\}$, and $C = \emptyset$.

Then $A \cap C = \emptyset \subseteq \emptyset = B \cap C$, but $A \not\subseteq B$.

7. For each $n \in \mathbb{N}$, let $A_n = [\frac{1}{n}, 2 + \frac{1}{n}) \subset \mathbb{R}$, and let $\mathcal{A} = \{A_n : n \geq 3\}$.
a. [5 pts.] What is $\bigcup_{n \geq 3} A_n$?

$(0, 2 + \frac{1}{3})$

b. [5 pts.] What is $\bigcap_{A \in \mathcal{A}} A$?

$[\frac{1}{3}, 2]$

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8. [20 pts.] In a certain kind of tournament, every player plays every other player exactly once and either wins or loses. There are no ties. Define a **top** player to be a player who, for every other player x, either beats x, or beats a player y who beats x.
Show that every n-player tournament has a top player.

The statement is trivially true for $n=1$ or $n=2$.

Assume that for some $n \geq 2$, every n-player tournament has a top player.

Now, consider any $(n+1)$-player tournament.

Choose one player s, and examine the play between all other players.

This is an n-player tournament, so there is a top player, t.

Let B be the set of all players that t beats.

Now, if t beats s, then t is a top player in the $(n+1)$-player tournament.

If not, then either some w in B beat s, or not.

If some w in B beat s, then t is a top player.

Otherwise, s beat t and every player in B, so s is a top player.

By PMI, $\forall n \in \mathbb{N}$, every n-player tournament has a top player.