

1. State the definition of each of the following terms:

a. [3 pts.] A **proposition**.

A proposition is a sentence that is either true or false.

b. [3 pts.] A **denial** of a given proposition.

A denial of a proposition  $P$  is any proposition equivalent to  $\sim P$ .

2. Determine whether each of the following is a tautology, a contradiction, or neither. Justify your response.

a. [5 pts.]  $[(P \Rightarrow Q) \Rightarrow P] \Rightarrow P$ .

P	Q	$P \Rightarrow Q$	$(P \Rightarrow Q) \Rightarrow P$	$[(P \Rightarrow Q) \Rightarrow P] \Rightarrow P$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Tautology

b. [5 pts.]  $(P \Rightarrow Q) \Leftrightarrow (P \wedge \sim Q)$ .

$$\begin{aligned} & (P \Rightarrow Q) \Leftrightarrow (P \wedge \sim Q) \\ \text{iff } & (Q \vee \sim P) \Leftrightarrow (P \wedge \sim Q) \\ \text{iff } & \sim(P \wedge \sim Q) \Leftrightarrow (P \wedge \sim Q) \end{aligned}$$

This is a **Contradiction** of the form  $R \wedge \sim R$ .

3. Which of the following are denials of  $(\exists!x)P(x)$ ? Justify your response.

a. [5 pts.]  $(\forall x)(\sim P(x)) \vee (\exists y)(\exists z)(y \neq z \wedge P(y) \wedge P(z))$ .

$$\begin{aligned} & \sim(\exists!x)P(x) \quad \text{iff } \sim[(\exists x)P(x) \wedge (\forall y)(\forall z)(P(y) \wedge P(z) \Rightarrow y=z)] \\ \text{iff } & [\sim(\exists x)P(x)] \vee [\sim(\forall y)(\forall z)(P(y) \wedge P(z) \Rightarrow y=z)] \quad \text{This is a } \boxed{\text{denial}} \text{ of} \\ \text{iff } & (\forall x)\sim P(x) \vee (\exists y)(\exists z)(P(y) \wedge P(z) \wedge y \neq z). \quad (\exists!x)P(x). \end{aligned}$$

b. [5 pts.]  $(\forall x)[P(x) \Rightarrow (\exists y)(P(y) \wedge x \neq y)]$ .

This sentence is true either when the truth set for  $P$  is empty or when the truth set of  $P$  is non-empty and there exist two distinct elements in the truth set. This is exactly when the sentence  $\sim(\exists!x)P(x)$  is true.

Since this sentence is equivalent to  $\sim(\exists!x)P(x)$ , it is a **denial** of  $(\exists!x)P(x)$ .

c. [5 pts.]  $\sim(\forall x)(\forall y)[(P(x) \wedge P(y)) \Rightarrow x=y]$ .

$$\begin{aligned} & \sim(\forall x)(\forall y)(P(x) \wedge P(y) \Rightarrow x=y) \\ \text{iff } & (\exists x)(\exists y)\sim(P(x) \wedge P(y) \Rightarrow x=y) \\ \text{iff } & (\exists x)(\exists y)(P(x) \wedge P(y) \wedge x \neq y). \end{aligned}$$

In a universe where  $P$  is never true, this sentence would be false, but  $\sim(\exists!x)P(x)$  would be true.

Hence, this sentence is **not a denial** of  $(\exists!x)P(x)$ .

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4. [15 pts.] Prove that for every pair of positive real numbers  $x$  and  $y$  where  $x < y$ , there exists a natural number  $M$  such that if  $n$  is a natural number and  $n > M$ , then  $\frac{1}{n} < (y - x)$ .

Let  $x$  and  $y$  be positive real numbers with  $x < y$ .

Then  $y - x > 0$ , so in particular,  $\frac{1}{y-x}$  is a positive real number.

Let  $M$  be one more than the integer part of  $\frac{1}{y-x}$ .

That is, pick  $M = \lfloor \frac{1}{y-x} \rfloor + 1$ . Then  $M \in \mathbb{N}$ .

And, for all natural numbers  $n$  s.t.  $n > M$ ,

$$n > M > \frac{1}{y-x}.$$

Since  $\frac{1}{y-x}$  is positive,  $\frac{1}{n} < (y-x)$ .

Thus,  $\forall x, y \in \mathbb{R}^+$ ,  $\exists M \in \mathbb{N}$  s.t.  $n > M, n \in \mathbb{N} \Rightarrow \frac{1}{n} < y-x$ .

5. a. [15 pts.] For  $x$  and  $y$  real numbers, show that if  $x+y$  is irrational, then either  $x$  is irrational or  $y$  is irrational. (Do not assume that  $\mathbb{Q}$  is closed under addition, but you may assume that  $\mathbb{Z}$  is.)

We prove this by Contrapositive.

Let  $x$  and  $y$  be real numbers.

Suppose that neither  $x$  nor  $y$  are irrational.

Then there exist integers  $p, q, r, s$ , with  $q \neq 0$  and  $s \neq 0$  s.t.

$$x = \frac{p}{q} \quad \text{and} \quad y = \frac{r}{s}.$$

$$\text{Then } x+y = \frac{p}{q} + \frac{r}{s} = \frac{ps}{qs} + \frac{rq}{qs} = \frac{ps+rq}{qs}.$$

Since  $\mathbb{Z}$  is closed under addition and multiplication,  $ps+rq$  and  $qs$  are integers.

And  $q \neq 0$  and  $s \neq 0$ , so  $qs \neq 0$ . Thus,  $x+y$  is rational.

Thus, by Contraposition, if  $x+y$  is irrational, then either  $x$  is irrational or  $y$  is irrational.

b. [5 pts.] Is the converse of the statement in 5.a. true? Justify your response.

No, the converse is false.

The converse is: If either  $x$  or  $y$  is irrational, then  $x+y$  is irrational.

But if  $x = \sqrt{2}$  and  $y = 6 - \sqrt{2}$ , then either  $x$  or  $y$  is irrational (since they both are), yet  $x+y = \sqrt{2} + 6 - \sqrt{2} = 6$  is rational.

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6. a. [15 pts.] For sets  $A$ ,  $B$ , and  $C$ , prove that if  $A \subset B$ , then  $A \cup C \subset B \cup C$ .

Let  $A$ ,  $B$ , and  $C$  be sets, and suppose  $A \subseteq B$ .

Now, suppose  $x \in A \cup C$ .

Then  $x \in A$  or  $x \in C$ .

If  $x \in C$ , then  $x \in B \cup C$ .

If  $x \in A$ , then since  $A \subseteq B$ ,  $x \in B$ , and  $x \in B \cup C$ .

Either way,  $x \in B \cup C$ .

Thus,  $A \cup C \subseteq B \cup C$ .

b. [5 pts.] Is the converse to the statement in 6.a. true? Justify your response.

The converse is false.

Consider the sets  $A = \{1\}$ ,  $B = \{2\}$ , and  $C = \{1, 2\}$ .

Then  $A \cup C \subseteq B \cup C$ , since  $A \cup C = B \cup C = \{1, 2\}$ .

However,  $A \not\subseteq B$ .

7. [20 pts.] For  $A$  and  $B$  sets, prove that  $A \subset B$  iff  $A - B = \emptyset$ .

Let  $A$  and  $B$  be sets.

Suppose, first, that  $A \subseteq B$ .

Now, if  $x \in A - B$ , then  $x \in A$  and  $x \notin B$ .

But since  $A \subseteq B$ ,  $x \in A \Rightarrow x \in B$ .

Therefore,  $A - B = \emptyset$ .

Now, suppose  $A - B = \emptyset$ .

If  $x \in A$  and  $x \notin B$ , then  $x \in A - B$ .

But  $A - B = \emptyset$ . Thus, for any  $x \in A$ ,  $x \in B$  also.

Thus,  $A \subseteq B$ .

Therefore,  $A \subseteq B$  iff  $A - B = \emptyset$ .

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8. Assign a grade of A (correct), C (partially correct), or F (failure) to each. Justify your assignments of grades.

a. [7 pts.] **Claim.**  $A \cap \bar{A} = \emptyset$ .

**“Proof”.** By Theorem 2.1(a),  $\emptyset$  is a subset of every set. Thus  $\emptyset \subset A \cap \bar{A}$ . Now, suppose  $x \in A \cap \bar{A}$ . Then  $x \in A$  and  $x \in \bar{A}$ . Thus  $x \in A$  and  $\sim(x \in A)$ . Therefore,  $x \neq x$ . Hence, by the definition of  $\emptyset$ ,  $x \notin \emptyset$ . Therefore,  $A \cap \bar{A} \subset \emptyset$ . Since the two sets are subsets of each other,  $A \cap \bar{A} = \emptyset$ .

C. This proof has the right idea, but it falsely claims that by the definition of  $\emptyset$ ,  $x \in \emptyset$ . The empty set contains no elements, so  $x \in \emptyset$  is impossible. Instead, one should conclude that  $(x \in A)$  and  $\sim(x \in A)$  is a contradiction, and thus the supposition that  $x \in A \cap \bar{A}$  was mistaken, and  $A \cap \bar{A}$  has ~~no~~ no elements.

b. [7 pts.] **Claim.**  $1^3 + 2^3 + \dots + n^3 = \left[ \frac{n(n+1)}{2} \right]^2$  for all natural numbers  $n$ .

**“Proof”.**  $1^3 = 1 = \left[ \frac{1(1+1)}{2} \right]^2$ , so the statement holds for 1. Assume the statement holds for some natural number  $n$ . Then,

$$1^3 + 2^3 + \dots + (n+1)^3 = \left[ \frac{(n+1)(n+2)}{2} \right]^2, \text{ so}$$

$$[1^3 + 2^3 + \dots + n^3] + (n+1)^3 = (n+1)^2 \left[ \frac{n^2 + 4n + 4}{4} \right], \text{ so}$$

$$\left[ \frac{n(n+1)}{2} \right]^2 + (n+1)^3 = (n+1)^2 \left[ \frac{n^2}{4} + n + 1 \right].$$

So the statement holds for  $n+1$ . Thus, by the Principle of Mathematical Induction, it is true for all natural numbers.

C. The proof assumes that the statement holds for  $n+1$  when it says  $1^3 + 2^3 + \dots + (n+1)^3 = \left[ \frac{(n+1)(n+2)}{2} \right]^2$ . Instead, it should start from one side of this equation, follow the same algebraic steps and manipulations, and arrive at the other side of the equation, concluding that they are equal.

9. For each  $n \in \mathbb{N}$ , let  $D_n = (-n, \frac{1}{n}) \subset \mathbb{R}$ , and let  $\mathcal{D} = \{D_n : n \in \mathbb{N}\}$ .  
 a. [5 pts.] What is  $\bigcup_{n \in \mathbb{N}} D_n$ ?

$$\bigcup_{n \in \mathbb{N}} D_n = (-\infty, 1).$$

- b. [5 pts.] What is  $\bigcap_{D \in \mathcal{D}} D$ ?

$$\bigcap_{D \in \mathcal{D}} D = (-1, 0].$$

10. [20 pts.] Let  $r$  be a real number different from 1. Prove that for all natural numbers  $n$ ,

$$\sum_{i=0}^{n-1} ar^i = \frac{a(r^n - 1)}{r - 1}.$$

We prove this by Induction on  $n$ .

For  $n=1$ ,  $\sum_{i=0}^{1-1} ar^i = \sum_{i=0}^0 ar^i = ar^0 = a = \frac{a(r^1 - 1)}{r - 1}$ ; so the statement holds for  $n=1$ .

Now, suppose the statement holds for some  $n \in \mathbb{N}$ .

Then,  $\sum_{i=0}^{(n+1)-1} ar^i = \sum_{i=0}^{n-1} ar^i + ar^n = \frac{a(r^n - 1)}{r - 1} + ar^n = \frac{a(r^n - 1)}{r - 1} + \frac{ar^n(r - 1)}{r - 1}$

By the Inductive Hypothesis

$$= \frac{a(r^n - 1) + ar^n(r - 1)}{r - 1} = \frac{a[r^n - 1 + r^n(r - 1)]}{r - 1}$$

$$= \frac{a[r^n - 1 + r^{n+1} - r^n]}{r - 1} = \frac{a(r^{n+1} - 1)}{r - 1}$$

So if the statement holds for some  $n \in \mathbb{N}$ , then it holds for  $n+1$ .

And by the Principle of Mathematical Induction, the statement holds for all  $n \in \mathbb{N}$ .