Homework 4

Exercises 3.4

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(a) $\{(a,a),(c,c)\}$

(b) $\{(a,a),(b,b),(b,c),(c,c),(a,c)\}$

(c) $\{(a,b),(a,c),(c,c)\}$

(d) $\{(a,b),(b,a),(b,b)\}$

(e) $\{(a,b),(b,a),(a,c),(c,c)\}$

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(i) For each $(x,y) \in \mathbb{R} \times \mathbb{R}$, $x \leq x$ and $y \leq y$, so $(x,y)R(x,y)$.

(ii) Suppose $(a,b)R(x,y)$ and $(x,y)R(a,b)$. Then $a \leq x$ and $x \leq a$, so $x = a$. Similarly, $y = b$. Therefore, $(a,b) = (x,y)$.

(iii) Suppose $(a,b)R(x,y)$ and $(x,y)R(c,d)$. Then $a \leq x \leq c$, so $a \leq c$ and $b \leq y \leq d$, so $b \leq d$. Therefore $(a,b)R(c,d)$.

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Claim 1. *Every subset of a well-ordered set is well ordered.*

Proof. Let A be any well-ordered set under the relation R. Let B be any subset of A. We first show that B is a linearly ordered set (under $R|_B$). For all $x \in B$, $x \in A$ as well. So xRx, which shows that the relation is reflexive on B. For any x, y, and z in B with xRy and yRz, again we note that x, y, and z are in A, which is linearly ordered, so xRz. Verifying that the relation is antisymmetric on B and that any two elements are comparable are similar to the above. So B is linearly ordered. It only remains to show that every nonempty subset of B has a least element. Suppose $C \subseteq B$ and $C \neq \emptyset$. Since $C \subseteq B \subseteq A$ we have $C \subseteq A$. But A is well-ordered, so every nonempty subset has a least element. Thus, C has a least element. Therefore B is well-ordered. □

Exercises 4.1

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(a) $D = \mathbb{R} \setminus \{-1\}$, $R = \mathbb{R} \setminus \{0\}$.

(h) $D = \mathbb{R}$, $R = [1, \infty)$.

(i) $D = \mathbb{R} \setminus \{2\}$, R $= \mathbb{R} \setminus \{4\}$.

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(e) $D = [3,5]$, $R = [\sqrt{2}, 2]$.

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(b) Function.

(d) Not a function. $\overline{0} = \overline{4}$ in $\mathbb{Z}_4$, but $f(\overline{0}) = [2 \cdot 0 + 1] = [1] \neq [3] = [9] = [2 \cdot 4 + 1] = f(\overline{4})$ in $\mathbb{Z}_6$.

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f and g are not equal as sets, because $(-3, 6) \in g$, but $(-3, 6) \notin f$. In particular, f and g have different domains.

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(b) A.

(c) A.

Exercises 4.2

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(b) $(f \circ g)(x) = 4x^2 + 8x + 3$; $(g \circ f)(x) = 2x^2 + 4x + 1$.

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Suppose h and g are functions such that $Dom(f) = A$, $Dom(g) = B$, and $A \cap B = \emptyset$. Then h and g are relations, so $h \cup g$ is a relation. If $x \in Dom(h \cup g)$, then there is a y such that $(x, y) \in h$ or $(x, y) \in g$, so $x \in A$ or $x \in B$. On the other hand, if $x \in A \cup B$, then there is a y such that $(x, y) \in h$ or $(x, y) \in g$, so $(x, y) \in h \cup g$, and so $x \in Dom(h \cup g)$. Thus, $Dom(h \cup g) = A \cup B)$. Now, suppose (x, y) and (x, z) are in $h \cup g$. By the preceding argument, $x \in A \cup B$, and since $A \cap B = \emptyset$, x cannot be in both A and B. If $x \in A$, then $y = h(x) = z$, since h is a function. If $x \in B$, then $y = g(x) = z$, since g is a function. Either way, $y = z$, and so $h \cup g$ is a function.

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(a) We first show that $f_1 + f_2$ is a function with domain $\mathbb{R}$. First, $f_1 + f_2$ is by definition a relation. For all $x \in \mathbb{R}$ there is some $u \in \mathbb{R}$ such that $(x, u) \in f_1$ because $f_1 : \mathbb{R} \to \mathbb{R}$ and there is some $v \in \mathbb{R}$ such that $(x, v) \in f_2$ because $f_2 : \mathbb{R} \to \mathbb{R}$. Then $(x, u + v) \in f_1 + f_2$, so $x \in \text{Dom}(f_1 + f_2) = \mathbb{R}$. It is clear from the definition of $f_1 + f_2$ that $x \in \text{Dom}(f_1 + f_2)$ implies $x \in \mathbb{R}$, so $\text{Dom}(f_1 + f_2) = \mathbb{R}$. Now let $x \in \mathbb{R}$. Suppose (x, c) and (x, d) are in $f_1 + f_2$. Then $c = f_1(x) + f_2(x) = d$. Therefore $f_1 + f_2$ is a function.

Now we show that $f_1 \cdot f_2$ is a function with domain $\mathbb{R}$. First, $f_1 \cdot f_2$ is a relation by definition. Now let $x \in \mathbb{R}$. Then there exist u and v in $\mathbb{R}$ such that $(x, u) \in f_1$ and $(x, v) \in f_2$, so $(x, uv) \in f_1 \cdot f_2$; thus, $x \in \text{Dom}(f_1 \cdot f_2)$. It is clear from the definition of $f_1 \cdot f_2$ that $\text{Dom}(f_1 \cdot f_2) \subseteq \mathbb{R}$; hence $\text{Dom}(f_1 \cdot f_2) = \mathbb{R}$. Now suppose (x, c) and (x, d) are both in $f_1 \cdot f_2$. Then $c = f_1(x) \cdot f_2(x) = d$, so $f_1 \cdot f_2$ is a function.

(b)

$$
\begin{array}{ll}
(f + g)(x) = 11 - 5x & (f \cdot g)(x) = -14x^2 - 23x + 30 \\
(f + h)(x) = 3x^2 - 5x + 7 & (g \cdot h)(x) = -21x^3 + 67x^2 - 56x + 12
\end{array}
$$

Exercises 4.3

1

(b) Onto. Let $n \in \mathbb{Z}$. Then $1000 - n \in \mathbb{Z}$ and $f(1000 - n) = -(1000 - n) + 1000 = n$.

(h) Onto. Let $x \in \mathbb{R}$. Then $(x, 0) \in \mathbb{R} \times \mathbb{R}$ and $f(x, 0) = x - 0 = x$.

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(b) One-to-one. Suppose $f(x) = f(y)$. Then $-x + 1000 = -y + 1000$, so $x = y$.

(h) Not one-to-one. Note that $f(1, 0) = f(2, 1)$.

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Suppose $f :\xrightarrow{\text{onto}} B$ and $g : B \xrightarrow{\text{onto}} C$. Let $c \in C$. Then there is $b \in B$ such that $g(b) = c$ since g is onto B. Also, there is an $a \in A$ such that $f(a) = b$ since f is onto A. Thus $(g \circ f)(a) = g(f(a)) = g(b) = c$, which proves that $g \circ f$ is onto.

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Claim 2. *If $f : A \to B$, $g : B \to C$, and $g \circ f : A \xrightarrow{\text{1-1}} C$, then $f : A \xrightarrow{\text{1-1}} B$.*

Proof. Suppose $f : A \to B$, $g : B \to C$, and $g \circ f : A \xrightarrow{\text{1-1}} C$. Suppose also that $x, y \in A$ and $f(x) = f(y)$. Since g is a function, $(g \circ f)(x) = g(f(x)) = g(f(y)) = (g \circ f)(y)$. Since $g \circ f$ is one-to-one, $x = y$. □

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(b) Let $A = \{1\}, B = \{1, 2\} = C, f = \{(1, 1)\}$ and $g = \{(1, 2), (2, 1)\}$.

(c) Let $A = \{1\}, B = \{1, 2\}, C = \{1\}, f = \{(1, 1)\}$ and $g = \{(1, 1), (2, 1)\}$.

(d) Let $A = \{1, 2\} = B, C = \{1\}, f = \{(1, 2), (2, 1)\}$ and $g = \{(1, 1), (2, 1)\}$.

(f) Let $A = \{1, 2\}, B = \{a, b, c\}, C = \{5, 6\}, f = \{(1, a), (2, b)\}$ and $g = \{(a, 5), (b, 6), (c, 6)\}$.