Homework 4

Exercises 3.1

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Claim 1. *For nonempty sets A and B, one has the identity $A \times B = B \times A$ if and only if $A = B$.*

Proof. ($\Rightarrow$) Suppose $A \times B = B \times A$. Let $b \in B$ and $a \in A$. Then $(a, b) \in A \times B = B \times A$; thus, $a \in B$ and $b \in A$. This shows $A \subseteq B$ and $B \subseteq A$; hence $A = B$.

($\Leftarrow$) If $A = B$, then by substitution, $A \times B = B \times A$.

□

The claim does not hold if A is empty and B is nonempty. In this case, $A \neq B$, even though $A \times B = \emptyset = B \times A$.

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(a) Let $A = \{1\}$, $B = C = \{3\}$, and $D = \{2\}$. Then

$$\begin{aligned} &(A \times B) \cup (C \times D) \text{ is the two-element set } \{(1,3), (3,2)\}, \text{ while} \\ &(A \cup C) \times (B \cup D) \text{ is the four-element set } \{(1,2), (1,3), (3,2), (3,3)\}, \end{aligned}$$

so $(A \times B) \cup (C \times D) \neq (A \cup C) \times (B \cup D)$.

(c) Let $A = B = C = \{1\}$. Then

$$A \times (B \times C) = \{(1, (1,1))\} \neq \{((1,1), 1)\} = (A \times B) \times C.$$

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Let T be the relation $\{(3,1), (2,3), (3,5), (2,2), (1,6), (2,6), (1,2)\}$.

(a) $\mathrm{Dom}(T) = \{1, 2, 3\}$.

(b) $\mathrm{Rng}(T) = \{1, 2, 3, 5, 6\}$.

(c) $T^{-1} = \{(1,3), (3,2), (5,3), (2,2), (6,1), (6,2), (2,1)\}$.
Note: Although the ordering within each pair is crucial, the order in which we list the collection of ordered pairs in T^{-1} does not matter. So we also have, for example

$$T^{-1} = \{(2,1), (6,2), (6,1), (2,2), (5,3), (3,2), (1,3)\},$$

since this is the same set as the one named above.

(d) By Theorem 3.3(a), $\left(T^{-1}\right)^{-1} = T = \{(3,1), (2,3), (3,5), (2,2), (1,6), (2,6), (1,2)\}$.

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Let R be a relation from A to B and let S be a relation from B to C.

(a) **Claim:** $\mathrm{Dom}(S \circ R) \subseteq \mathrm{Dom}(R)$. **Proof:** Suppose $x \in \mathrm{Dom}(S \circ R)$. Then for some z, $(x, z) \in S \circ R$. Then for some y, $(x, y) \in R$ and $(y, z) \in S$. Since $(x, y) \in R$, we have $x \in \mathrm{Dom}(R)$.

(c) The statement $\mathrm{Rng}(S \circ R) \subseteq \mathrm{Rng}(S)$ is always true. This may be proved as follows. Suppose $z \in \mathrm{Rng}(S \circ R)$. Then for some x, $(x, z) \in S \circ R$. Then for some y, $(x, y) \in R$ and $(y, z) \in S$. Since $(y, z) \in S$, we have $z \in \mathrm{Rng}(S)$.
On the other hand, if $R = \{(a, c)\}$ and $S = \{(b, d)\}$, then $\mathrm{Rng}(S) = \{d\} \nsubseteq \emptyset = \mathrm{Rng}(S \circ R)$.

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(b) Grade: C. This "proof" could be corrected by observing that $(a, c) \notin B \times D$ implies $a \notin B$ or $c \notin D$, which implies $a \notin A$ or $c \notin C$. This contradicts the fact that $a \in A$ and $c \in C$. A direct proof would be easier.

(c) Grade: F. The symbols

$$\frac{A \times B}{A} \text{ and } \frac{A \times C}{A}$$

seem to suggest "division" by the set A, but this operation is not defined.

(d) Grade: A.

Exercises 3.2

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(b) reflexive, transitive

(f) symmetric

(i) symmetric

(j) symmetric

(l) reflexive, transitive

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(a)

Claim 2. *The relation R on $\mathbb{Z}$ given by $x\,R\,y$ iff $x^2 = y^2$ is an equivalence relation.*

Proof. The relation R satisfies the properties of an equivalence relation, as follows.

1. Reflexive: Let $x \in \mathbb{Z}$. Then $x^2 = x^2$, so $x\,R\,x$.
2. Symmetric: Suppose $x\,R\,y$ for some $x, y \in \mathbb{Z}$. Then $x^2 = y^2$, so $y^2 = x^2$. Thus, $y\,R\,x$.
3. Transitive: Suppose $x\,R\,y$ and $y\,R\,z$ for some $x, y, z \in \mathbb{Z}$. Then $x^2 = y^2$ and $y^2 = z^2$, so $x^2 = z^2$.

Thus, R is an equivalence relation on $\mathbb{Z}$. □

The equivalence class of 0 determined by R (denoted $0/R$) is $\{0\}$, while $4/R = \{\pm 4\}$, and $-72/R = \{\pm 72\}$.

(c)

Claim 3. *The relation V on $\mathbb{R}$ given by $x\,V\,y$ iff $x = y$ or $xy = 1$ is an equivalence relation.*

Proof. The relation V satisfies the properties of an equivalence relation, as follows.

1. Reflexive: Let $x \in \mathbb{R}$. Then $x = x$, so $x\,V\,x$.
2. Symmetric: Suppose $x\,R\,y$ for some $x, y \in \mathbb{Z}$. Then one of the following is true: $x = y$, so $y = x$, or $xy = 1$, in which case $yx = 1$. Thus, $y\,V\,x$.

3. Transitive: Suppose $x\,R\,y$ and $y\,R\,z$ for some $x, y, z \in \mathbb{Z}$. Then $x = y$ or $xy = 1$, and $y = z$ or $yz = 1$. If $y = z$ then $x = z$ or $xz = 1$. If $yz = 1$, then $xz = 1$ (in the case $x = y$) or $x = z$ (when $xy = 1$, by cancellation in $xy = yz = 1$). In any case $x = z$ or $xz = 1$, so $x\,V\,z$.

Thus, R is an equivalence relation on $\mathbb{Z}$. □

We have $3/V = \left\{\frac{1}{3}, 3\right\}$, $\left(-\frac{2}{3}\right)/V = \left\{-\frac{2}{3}, -\frac{3}{2}\right\}$, and $0/V = \{0\}$.

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(b) There are eight equivalence classes for the relation of congruence modulo 8. They are the following.

$$\begin{array}{ll} 0/\equiv_8 = \{\ldots, -16, -8, 0, 8, \ldots\} & 4/\equiv_8 = \{\ldots, -12, -4, 4, 12, \ldots\} \\ 1/\equiv_8 = \{\ldots, -15, -7, 1, 9, \ldots\} & 5/\equiv_8 = \{\ldots, -11, -3, 5, 13, \ldots\} \\ 2/\equiv_8 = \{\ldots, -14, -6, 2, 10, \ldots\} & 6/\equiv_8 = \{\ldots, -10, -2, 6, 14, \ldots\} \\ 3/\equiv_8 = \{\ldots, -13, -5, 3, 11, \ldots\} & 7/\equiv_8 = \{\ldots, -9, -1, 7, 15, \ldots\} \end{array}$$

(c) There is one equivalence class for the relation of congruence modulo 1, namely, $0/\equiv_1 = \mathbb{Z}$.

Exercises 3.3

2

(a) The collection $\mathcal{A}$ is not a partition of A, since $\mathcal{A}$ is not pairwise disjoint. That is, it fails criterion (ii) of the definition on p. 154 of the text.

(b) The collection $\mathcal{A}$ is not a partition of A, since $\mathcal{A}$ does not cover A, i.e., it fails criterion (iii) of the definition, since

$$\bigcup_{X \in \mathcal{A}} X = \{1, 2, 3, 4, 5\} \neq \{1, 2, 3, 4, 5, 6, 7\} = A.$$

(c) The collection $\mathcal{A}$ is a partition of A.

6(a)

The equivalence relation on $\mathbb{N}$ with the given partition is

$$x\,R\,y \text{ iff } 2^n \leq x, y < 2^{n+1} \text{ for some } n \in \mathbb{N}.$$

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(Completing the proof of Theorem 3.6) If $\mathcal{B}$ is a partition of A, and Q is a relation such that $x\,Q\,y$ iff there exists $C \in \mathcal{B}$ such that $x \in C$ and $y \in C$, then

(a) Q is symmetric.
Proof: Suppose $x\,Q\,y$. By definition of Q, there is $C \in \mathcal{B}$ such that $x \in C$ and $y \in C$. Since both y and x belong to C, $y\,Q\,x$. Therefore, Q is symmetric.

(b) Q is reflexive on A.
Proof: Let $t \in A$. Since $\mathcal{B}$ is a partition of A, we have $A = \cup_{X \in \mathcal{B}} X$. Consequently, there is some $C \in \mathcal{B}$ so that $t \in C$. Thus $t\,Q\,t$. Therefore, Q is reflexive on A.