Homework 3

Exercises 2.5

3(a)

Claim 1. *If $a > 0$, then for every natural number n, $a^n > 0$.*

Proof. Let $a > 0$ and $T = \{n \in \mathbb{N} : a^n \leq 0\}$. Suppose $T \neq \emptyset$. Then by the WOP, T has a smallest element, n. Thus, $n - 1 \notin T$, so $a^{n-1} > 0$. Therefore, because the product of two positive integers is positive, $a^n = a \cdot a^{n-1} > 0$. This contradicts the fact that $n \in T$. We conclude that T is empty, so $a^n > 0$ for all $n \in \mathbb{N}$. □

3(b)

Claim 2. *For all positive integers a and b, $b \neq a + b$.*

Proof. Suppose there were natural numbers a, b such that $b = a + b$. Then by the WOP, the set $T = \{n \in \mathbb{N} : n = a + n\}$ has a least element, b_0. And $b_0 \neq 1$ because 1 is not the successor of any natural number. Now, $b_0 = a + b_0$ implies $(b_0 - 1) = a + (b_0 - 1)$, which implies $b_0 - 1 \in T$. This contradicts the minimality of b_0. Therefore, T is empty. Since a is arbitrary, we conclude that for $a, b \in \mathbb{N}$, $b \neq a + b$. □

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Theorem (Archimedean Principle)**.** *For all natural numbers a and b, there exists a natural number s such that $a < sb$.*

Proof. We use induction on a.

(i) If $a = 1$, choose $s = 2$. Then
$$a = 1 \leq b < b + b = 2b = s \cdot b.$$

(ii) Suppose the statement is true when $a = n$ for every natural number b, and let $b \in \mathbb{N}$. Then there is an $s \in \mathbb{N}$ such that $n < s \cdot b$, so $s + 1 \in \mathbb{N}$ and
$$\begin{aligned} n + 1 &< s \cdot b + 1 \\ &\leq s \cdot b + b \\ &= (s+1)\,b. \end{aligned}$$
Thus, the statement is true for $a = n + 1$.

(iii) By the PMI, the statement is true for all natural numbers a and b.

□

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(a) Grade: A.

(b) Grade: F. The "proof" fails to consider that a special argument is needed for the case $m = 1$. In this case, $m - 1$ is not a natural number. The claim is false.

(c) Grade: F. One cannot know that $\{1, 2, \ldots, x - 1\} \subseteq \mathbb{N} - T$ unless one knows (or assumes) that x is the smallest element of T.

(d) Grade: F. The characterization of S does not match the assumption made in the PCI.